Researches on Rough Truth of Rough Axioms Based on Granular Computing

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***Abstract*—The concept, rough truth, is first presented in rough logic. It is a logical value, and lies between truth and falsity. By combining rough logic with modal logic, rough validity of the rough axioms is studied in this paper, which has close links with the logical value: rough truth. Because an axiom of modal logic corresponds to a rough axiom, the study of this paper actually focuses on the analysis of rough truth of the axioms in modal logic, which is based on a structure constructed in this paper. The structure is linked with a series of special states. The research on rough truth connects the special states with the rough axioms. At the same time, granular computing is introduced to the research process. As an approach to data processing, granular computing plays an important role in determining whether a rough axiom is roughly true or not. Thus, the study also demonstrates a way of research on granular computing. The conclusions show that each rough axiom is roughly true at every state of the structure, which means that each rough axiom is roughly valid. This is the desired result.**

***Index Terms*—rough truth, rough validity, rough axiom, granular computing, operator**

I. INTRODUCTION

Modal logic [1][2] is an extension of classical logic. In it, the operators □ and ◇ appear in formulas, in particular, in axioms, which does not occur in classical logic. Naturally, a formal system of modal logic is an extension of a formal system in classical logic because of the occurrence of □ and ◇. Meanwhile, the operators □ and ◇ lead to the emergence of a structure called a Kripke model [2] in which the meaning of □ or ◇ can be interpreted. Accordingly, the formulas involving □ or ◇ have their semantics. Although the Kripke model is different from a semantic structure of classical logic, the formulas in modal logic still take truth or falsity as the logical values, which have the same situation as classical formulas. So, it is worthwhile to discuss the subject that formulas of modal logic take other logical values situated between truth and falsity. This is what we are going to study in this paper.

Now recall rough logic [3] in which Z. Pawlak, the originator of rough set theory, introduced five logical values: truth, falsity, rough truth, rough falsity and rough inconsistency. Among them, rough truth is a logical value that lies between truth and falsity, and does not occur in modal logic. The investigation on rough truth about the formulas of modal logic will be what we discuss in the following. Actually, the developments in [4] and [5] have integrated rough sets with modal logic. The former mainly study the connection between the axioms of modal logic and algebraic properties of binary relations, but it is not concerned with rough truth. The latter creates a special system of modal logic based on an incomplete information system, but the special system is linked with the traditional method, its formulas still take truth or falsity as the logical values. Thus, the analysis on rough truth about logical formulas, in particular, about the axioms of modal logic will be an interesting subject that will be studied in this paper. We will concentrate our attention on rough truth of the formulas defined in this paper, especially on rough truth about the rough axioms which correspond to the axioms in modal logic.

With this end in view, we will integrate modal logic with rough logic, which will correlate with new operators. The new operators being similar to the modal operators □ and ◇ will be connected with rough truth. According to the general steps in mathematical logic, we need to define formulas. This reminds us of the formulas in modal logic that involve the operators □ or ◇, and makes us notice the formulas in rough logic that are based on an information system $S=(D, A, V, f)$[3]. So we have the idea of integrating the two sorts of the formulas. What we do will follow this idea. The formulas in this paper will extend the formulas in rough logic, also will involve new operators. Making use of the formulas, we will define granules, and explain what is granular computing. This enables us to establish a connection between rough truth and the axioms in modal logic. Thus, formulas will be an important basis of the following discussion. We now start from the definition of formulas.

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II. Formulas and Rough Truth

A. Construction of Formulas

The formulas in this paper will be an extension of the formulas in rough logic. Also, the formulas will involve some operators as in modal logic. From [3] we know that the formulas of rough logic are based on an information system $S=(D, A, V, f)$, where $D=\{u_1,\ldots, u_m\}$ is a finite set call the universal set; $A=\{a_1,\ldots, a_n\}$ is the attribute set, each element of A is referred to as an attribute; V is also a finite set which is the range of f, an element of V is called a value; f is a function from $D\times A$ to V, such that for $<u, a>\in D\times A$, there is a unique value $v\in V$, satisfying $f(u, a)=v$. Generally, the expression $f(u, a)=v$ is abbreviated to $a(u)=v$. Thus, each attribute $a(\in A)$ is actually a function from D to V.

Let $S=(D, A, V, f)$ be an information system. We now recall the formulas in rough logic. If $a\in A$ and $v\in V$, the notation (a, v) is called an atomic formula [3] in rough logic. By logical connectives, other formulas based on atomic formulas can be obtained, such as $(a_1, v_1)\wedge(a_2, v_2)$, $(a_1, v_1)\vee(a_2, v_2)$, $(a_1, v_1)\rightarrow(a_2, v_2)$ etc. are formulas [3] in rough logic, where $a_1, a_2\in A$, and $v_1, v_2\in V$.

The formulas in this paper will be linked with U that is a finite set called a universal set. We use $U^n(n\geqslant 1)$ to stand for $U\times\ldots\times U$, the Cartesian product of n factors of U. An element of U^n is denoted by $<u_1,\ldots, u_n>$, where $u_i\in U(i=1,\ldots, n)$. For a natural number $m(\geqslant 1)$, if H is a subset of U^m, i.e. $H\subseteq U^m$, then H is called an m-place relation on U. We will use m-place relations to define formulas that will be the basis for our study.

Either in rough logic or in modal logic, formulas are based on a symbol system. In order to define formulas, a symbol system needs to be introduced.

Definition 1. Let U be a universal set. The symbol system on U is defined by the following:

1) Constant: if $u\in U$, u is called a constant on U.
2) Variable: $x_1, x_2, x_3,\ldots$ denote variables on U.
3) Term: constants and variables are called terms on U, and $t_1, t_2, t_3,\ldots$ are used to stand for any term.
4) Relation: P, Q, S, H etc. or $P_1, P_2, P_3,\ldots$ denote any m-place relation on $U(m\geqslant 1)$.
5) Logical connectives: $\neg, \wedge, \vee, \rightarrow$.
6) Punctuation: the symbols " (", ") " and " , " are used as punctuation marks. ▌

Using the symbol system on U, we can define atomic formulas or formulas.

Definition 2. Let U be a universal set, and P be an m-place relation on $U(m\geqslant 1)$. If $t_1,\ldots, t_m$ are m terms, then $P(t_1,\ldots, t_m)$ is called an atomic formula on U. ▌

For example, $P(x_1,\ldots, x_n, u_1,\ldots, u_{m-n})(n\leqslant m)$ is an atomic formula on U, where $x_1,\ldots, x_n$ are variables, $u_1,\ldots, u_{m-n}(\in U)$ are constants, they are all terms, and $P(\subseteq U^m)$ is an m-place relation on U.

In modal logic, there are two modal operators $\Box$ and $\Diamond$ expressing "necessity" and "possibility" respectively. Although the formulas involving $\Box$ or $\Diamond$ are interpreted in Kripke models [2], the formulas still take truth or falsity as the logical values. Thus research on rough truth of formulas will be different from the study in modal logic. To this end, we intend to connect rough truth with special formulas which are similar to the axioms in modal logic. To get the special formulas, we introduce two operators $\blacksquare$ and $\blacklozenge$ corresponding to $\Box$ and $\Diamond$ respectively. The semantics of the new operators will be given in definition 6. The purpose of introducing $\blacksquare$ and $\blacklozenge$ is to define formulas.

Definition 3. Let U be a universal set. The formulas on U are inductively defined as follows:

1) Every atomic formula $P(t_1,\ldots, t_n)$ is a formula on U.
2) If φ is a formula on U, then $\neg\varphi$, $\blacksquare\varphi$ and $\blacklozenge\varphi$ are formulas on U.
3) If φ and ψ are formulas on U, then $\varphi\wedge\psi$, $\varphi\vee\psi$ and $\varphi\rightarrow\psi$ are formulas on U.
4) Formulas on U are generated by using 1), 2) or 3) in finite steps. ▌

Definition 2 shows that an m-place relation$(m\geqslant 1)$ determines an atomic formula. Also, from definition 3, we know that formulas on U are based on atomic formulas. Thus each formula on U has close links with m-place relations. Since there may exist variables in each atomic formula, any formula may also contain variables. If there are n variables $x_1,\ldots, x_n$ in a formula φ, then φ is called an n-place formula, and is also denoted by $\varphi(x_1,\ldots, x_n)$ in order to stress the n variables.

B. Semantic Sets of Formulas

For every formula on U, we will define its logical values by introducing the semantic set of the formula. In fact, rough logic [3] is concerned with this concept which can be briefly described in the following paragraph:

Let $S=(D, A, V, f)$ be an information system. The notation (a, v) is called an atomic formula in rough logic, where $a\in A$ and $v\in V$. For $u\in D$, if $a(u)=v$, we say that u satisfies (a, v). Let $|(a, v)|=\{u \mid u\in D$, and u satisfies $(a, v)\}$ that is subset of D, and consists of the elements satisfying (a, v). In rough logic, the set $|(a, v)|$ is referred to as the meaning of (a, v). In addition, for formulas $(a_1, v_1)\wedge(a_2, v_2)$, $(a_1, v_1)\vee(a_2, v_2)$ and $(a_1, v_1)\rightarrow(a_2, v_2)$, their meanings are defined by $|(a_1, v_1)\wedge(a_2, v_2)|=|(a_1, v_1)|\cap|(a_2, v_2)|$, $|(a_1, v_1)\vee(a_2, v_2)|=|(a_1, v_1)|\cup|(a_2, v_2)|$, and $|(a_1, v_1)\rightarrow(a_2, v_2)|=\sim|(a_1, v_1)|\cup|(a_2, v_2)|$, respectively. Thus, every formula in rough logic has its meaning.

Being similar to the meaning of a formula in rough logic, the semantic set of some formulas on U can be defined. Consider the following discussion.

Let $P\subseteq U^m$, and let $P(x_1,\ldots, x_n, u_1,\ldots, u_{m-n})$ be an n-place atomic formula on U, where $x_1,\ldots, x_n$ are variables, $u_1,\ldots, u_{m-n}$ are constants, and $1\leqslant n\leqslant m$. For $<t_1,\ldots, t_n>\in U^n$, if $<t_1,\ldots, t_n, u_1,\ldots, u_{m-n}>\in P$, then we say that $<t_1,\ldots, t_n>$ satisfies $P(x_1,\ldots, x_n, u_1,\ldots, u_{m-n})$. When $n=1$, $<t_1>$ is simply denoted by t_1.

For an n-place atomic formula $P(x_1,\ldots, x_n, u_1,\ldots, u_{m-n})$ $(n\geqslant 1)$, let us consider the set $\{<t_1,\ldots, t_n> \mid <t_1,\ldots, t_n>\in U^n$, and $<t_1,\ldots, t_n>$ satisfies $P(x_1,\ldots, x_n, u_1,\ldots, u_{m-n})\}$ which is denoted by $|P(x_1,\ldots, x_n, u_1,\ldots, u_{m-n})|$. Clearly, it is a subset of U^n, i.e. $|P(x_1,\dots, x_n, u_1,\dots, u_{m-n})|\subseteq U^n$. We refer to $|P(x_1,\dots, x_n, u_1,\dots, u_{m-n})|$ as the semantic set of the atomic formula $P(x_1,\dots, x_n, u_1,\dots, u_{m-n})$.

In addition atomic formulas, the semantic set is also linked with other formulas. Consider the formulas which do not involve the operators ■ and ◆. If φ is such a formula, we use $|\varphi|$ to denote the semantic set of φ. The next definition makes it clear.

Definition 4. Let φ and ψ be n-place formulas on U ($n \geqslant 1$). The operators ■ and ◆ do not occur in φ and ψ. The semantic set determined by each of the formulas that do not involve the operators ■ and ◆ are inductively defined as follows:

1) $|\neg\varphi| = \sim|\varphi|$ $(=U^n-|\varphi|)$.
2) $|\varphi\wedge\psi| = |\varphi|\cap|\psi|$.
3) $|\varphi\vee\psi| = |\varphi|\cup|\psi|$.
4) $|\varphi\rightarrow\psi| = |\neg\varphi|\cup|\psi|$. ▮

This definition shows that if φ and ψ are n-place formulas on U, and do not involve the operators ■ and ◆, then the semantic sets $|\neg\varphi|$, $|\varphi\wedge\psi|$, $|\varphi\vee\psi|$ and $|\varphi\rightarrow\psi|$ are subsets of U^n, i.e. $|\neg\varphi|\subseteq U^n$, $|\varphi\wedge\psi|\subseteq U^n$, $|\varphi\vee\psi|\subseteq U^n$ and $|\varphi\rightarrow\psi|\subseteq U^n$. Besides, when $<t_1,\dots, t_n>\in|\varphi|$, we say that $<t_1,\dots, t_n>$ satisfies φ. So, $|\varphi|=\{<t_1,\dots, t_n> \mid <t_1,\dots, t_n>\in U^n$, and $<t_1,\dots, t_n>$ satisfies $\varphi\}$, i.e. $|\varphi|$ consists of the elements that satisfy φ.

C. Further Explanation for Formulas

First, we are sure that the formulas in definition 3 are an extension of the formulas in rough logic. In fact, consider an information system $S=(D, A, V, f)$. Let (a, v) be an atomic formula in rough logic, where $a\in A$ and $v\in V$. Since $a\,(\in A)$ is a function from D to V, it can determine a binary-relation, say P_a, on U, where $U=D\cup V$, such that for $<u, v>\in D\times V$(of course $<u, v>\in U\times U$), $<u, v>\in P_a$ if and only if $a(u)=v$. Thus, we can get $P_a(x_1, v)$ that is a unary-place atomic formula on U, where x_1 is a variable, and $v(\in U)$ is a constant. For $u\in D$(of course $u\in U$), u satisfies $P_a(x_1, v)$ if and only if $<u, v>\in P_a$ if and only if $a(u)=v$, if and only if u satisfies (a, v). This indicates that the atomic formula (a, v) is actually the unary-place atomic formula $P_a(x_1, v)$. Since binary-place relations are a special case of m-place relations, the atomic formulas in definition 2 extend the atomic formulas in rough logic. Notice that the formulas in definition 3 are based on atomic formulas. Hence, the formulas in this paper are an extension of the formulas in rough logic. The formulas on U cover a wider range.

Second, in some cases, an n-place formula on U can become an m-place formula on U, where $m>n$, or $m<n$. Let us consider the following explanation:

Let $\varphi(x_1,\dots, x_n)$ be an n-place formula($n\geqslant 1$), and the operators ■ and ◆ do not occur in $\varphi(x_1,\dots, x_n)$. From the paragraph below definition 4, we know that $|\varphi(x_1,\dots, x_n)|=\{<t_1,\dots, t_n> \mid <t_1,\dots, t_n>\in U^n$, and $<t_1,\dots, t_n>$ satisfies $\varphi(x_1,\dots, x_n)\}$. For a natural number m and $m>n$, let $|\varphi(x_1,\dots, x_n)| =\{<t_1,\dots, t_n, t_{n+1},\dots, t_m> \mid <t_1,\dots, t_n, t_{n+1},\dots, t_m>\in U^m$, $<t_1,\dots, t_n>$ satisfies $\varphi(x_1,\dots, x_n)$, and $t_{n+1},\dots, t_m\in U\}$, then $|\varphi(x_1,\dots, x_n)|$ is a subset of U^m, i.e. $|\varphi(x_1,\dots, x_n)|\subseteq U^m$. In this case, $\varphi(x_1,\dots, x_n)$ can be viewed as an m-place formula and $m>n$. On the other hand, for a natural number m and $m<n$, select the constant elements $u_{m+1},\cdots, u_n\in U$, we therefore get the m-place formula $\varphi(x_1,\dots, x_m, u_{m+1},\cdots, u_n)$ and $|\varphi(x_1,\dots, x_m, u_{m+1},\cdots, u_n)|\subseteq U^m$. In accordance with this way, an n-place formula φ can be regarded as an m-place formula, where $m>n$ or $m<n$. So, if φ is an n-place formula, ψ is an m-place formula, and $n\neq m$, then φ and ψ can all be taken as either n-place or m-place formulas. Thus, the semantic sets $|\varphi\wedge\psi| = |\varphi|\cap|\psi|$, $|\varphi\vee\psi| = |\varphi|\cup|\psi|$ and $|\varphi\rightarrow\psi| = |\neg\varphi|\cup|\psi|$ may be either a subset of U^n, or a subset of U^m, which depends on the need of discussion. Generally, if $\varphi(x_1,\dots, x_n)$ is an n-place formula on U, it is natural that $|\varphi(x_1,\dots, x_n)|$ is a subset of U^n, i.e. $|\varphi(x_1,\dots, x_n)|\subseteq U^n$.

D. Rough Truth of Formulas

We know that rough logic is based on an information system $S=(D, A, V, f)$. Rough truth is a logical value defined in rough logic. It is connected with a structure (D, R) called an approximation space [6], where R is an equivalence relation on D, and is determined by some attributes of A. In this paper, rough truth will also correlate an approximation space which is somewhat different from (D, R). The approximation space is composed of U^n and R, denoted by (U^n, R), where U is a universal set; R is an equivalence relation on $U^n(n\geqslant 1)$. Now we explain the reason why U^n, rather than U, occurs in the approximation space (U^n, R):

Let $\varphi(x_1,\dots, x_{n-1}, x_n)$ be an n-place formula on U. It follows from definition 4 that $|\varphi(x_1,\dots, x_{n-1}, x_n)|$ is a subset of U^n, i.e. $|\varphi(x_1,\dots, x_{n-1}, x_n)|\subseteq U^n$. Let $\psi=\varphi(x_1,\dots, x_{n-1}, u)$, where $u\in U$, and u is a constant. It is clear that ψ is an $(n-1)$-place formula, and $|\psi|\subseteq U^{n-1}$. Thus, the objects in a semantic set not only correlate with U^n, but also with U. In this paper, rough truth will be defined by both a semantic set and an approximation space. Space (U^n, R) will be a suitable structure for the definition.

Let $M=(U^n, R)$ be an approximation space, and $U^n/R=\{[\boldsymbol{b}_1],\dots, [\boldsymbol{b}_t]\}$ be the partition of U^n, where $\boldsymbol{b}_i=<t_1,\dots, t_n>\in U^n$, and $[\boldsymbol{b}_i]=\{\boldsymbol{w}\mid \boldsymbol{w}=<t_1,\dots, t_n>\in U^n$ and $<\boldsymbol{b}_i, \boldsymbol{w}>\in R\}$ which is referred to as an equivalence class ($i=1,\dots, t$). And we call U^n/R the partition of U^n relative to R. When $X\subseteq U^n$, R-upper approximation $R^*(X)$ [6], and R-lower approximation $R_*(X)$ [6] about X in rough set theory are defined by the following expressions:

$R^*(X)=\cup\{[\boldsymbol{b}_i]\mid[\boldsymbol{b}_i]\in U^n/R$ and $[\boldsymbol{b}_i]\cap X\neq\varnothing\}$,

$R_*(X)=\cup\{[\boldsymbol{b}_i]\mid[\boldsymbol{b}_i]\in U^n/R$ and $[\boldsymbol{b}_i]\subseteq X\}$.

Where the notation "$\cup\{A \mid A$ is a set, and …}" expresses the union of the elements in $\{A\mid A$ is a set, and …}. For example, $\cup\{A, B, C\}=A\cup B\cup C$. Hence, if $X\subseteq U^n$, then $R^*(X)\subseteq U^n$ and $R_*(X)\subseteq U^n$. Also, by the definition of R-upper and R-lower approximations about X, we have $R_*(X)\subseteq X\subseteq R^*(X)$.

Now we use the semantic set of a formula to define logical values of the formula. From next definition we will notice that upper approximation and a semantic set are linked together, which will lead to the definition of rough truth.

Definition 5. Let (U^n, R) be an approximation space, and φ be an n-place formula on U. If the operators ■ and ◆ do not occur in φ, then:

1) If $|\varphi|=U^n$, then φ is said to be true in (U^n, R), denoted by $U^n \models \varphi$.

2) If $|\varphi| \neq U^n$, then φ is said to be false in (U^n, R), denoted by $U^n \not\models \varphi$.

3) If $R^*(|\varphi|)=U^n$, then φ is said to be roughly true in (U^n, R), denoted by $(U^n, R) \models \varphi$. ▮

This definition shows that rough truth of φ, i.e. φ is roughly true in (U^n, R), is determined by R-upper approximation about the semantic set $|\varphi|$ that is subset of U^n. Thus, rough truth of φ is relevant to both U^n and R which forms the approximation space (U^n, R). But truth and falsity of φ are only related to U^n, not correlating with the equivalence relation R.

Since $|\varphi| \subseteq R^*(|\varphi|) \subseteq U^n$, it is easy to know that $|\varphi|=U^n$ implies $R^*(|\varphi|)=U^n$. Thus, if φ is true in (U^n, R), then φ must be roughly true in (U^n, R). But, $R^*(|\varphi|)=U^n$ cannot guarantee $|\varphi|=U^n$. This illustrates that rough truth is weaker than truth. On the other hand, it is not difficult to know that falsity is weaker than rough truth, which means that we can find a formula φ, such that φ is false in (U^n, R), i.e. $|\varphi| \neq U^n$, but φ is not roughly true in (U^n, R), i.e. $R^*(|\varphi|) \neq U^n$. Thus rough truth lies between truth and falsity. Besides, there is a formula which is true and roughly true in (U^n, R). Also, there is a formula which is roughly true and false in (U^n, R). These characteristics are different from the truth value of a formula in classical logic and in other non-classical logics, because the logical values in definition 5 are determined by different cases of the semantic set $|\varphi|$ which is a subset of U^n. We will give a definition in section 3 which shows that the semantic set $|\varphi|$ and R-upper approximation $R^*(|\varphi|)$ are all granules. Hence the logical values in definition 5 are defined by different cases of granules.

To discuss the semantics of the operators ■ and ◆, we consider a set denoted by R^n, such that $R^n=\{R \mid R$ is an equivalence relation on $U^n\}$, where $n \geqslant 1$. Also, consider a another set denoted by P^n, defined as $P^n=\{(U^n, R) \mid R \in R^n\}$. Let $\subseteq$ be the relation of set containment. Because $\subseteq$ is reflective, anti-symmetric and transitive, $\subseteq$ is a partial order and $(R^n, \subseteq)$ is a partially ordered set. Using the relation $\subseteq$, we define a relation on P^n, denoted by $\leqslant$ which is also a partial order, satisfying the condition: for $(U^n, R_1), (U^n, R_2) \in P^n$, $(U^n, R_1) \leqslant (U^n, R_2)$ if and only if $R_1 \subseteq R_2$. Obviously, $(P^n, \leqslant)$ is a partially ordered set. Since P^n is uniquely determined by U and n, and is linked with U^n, we call $(P^n, \leqslant)$ the structure on U^n.

The structure $(P^n, \leqslant)$ can be regarded as an extension of an information system $S=(D, A, V, f)$. In fact, let us consider its attribute set $A=\{a_1,\ldots, a_n\}$. For each $a_i \in A$, since a_i is a function from D to $V(i=1,\ldots, n)$, a_i can determine an equivalence relation on D, denoted by R_{a_i}, such that $<u_j, u_k> \in R_{a_i}$ if and only if $a_i(u_j)= a_i(u_k)$ for $u_j, u_k \in D$. We therefore get an approximation space (D, R_{a_i}) which uniquely corresponds to the attribute a_i. Let $<(D, R_{a_1}),\ldots, (D, R_{a_n})>$ be the structure composed of the approximation spaces which correspond to the attributes of A. Information system $S=(D, A, V, f)$ is closely linked to $<(D, R_{a_1}),\ldots, (D, R_{a_n})>$. So, we can base rough logic on the structure $<(D, R_{a_1}),\ldots, (D, R_{a_n})>$. Now, let $U=D \cup V$. Consider $(P^n, \leqslant)$ which is the structure on $U^n(n \geqslant 1)$. Since P^n consists of all the approximation spaces on U^n, $(P^n, \leqslant)$ can be viewed as an extension of $<(D, R_{a_1}),\ldots, (D, R_{a_n})>$. The relation $\leqslant$ in $(P^n, \leqslant)$ is a concept not occurred in $<(D, R_{a_1}),\ldots, (D, R_{a_n})>$. It establishes connections between the elements of P^n.

The structure $(P^n, \leqslant)$ will be used as a model in which the semantics of the operators ■ and ◆ will be interpreted.

Note that in definition 5 we use $(U^n, R) \models \varphi$ to denote rough truth of φ in (U^n, R), where φ does not involve the operators ■ and ◆, of course, φ may be an atomic formula. Thus, if the operators ■ and ◆ do not occur in formula φ, $(U^n, R) \models \varphi$ has been defined already. This is the basis for the next definition.

Definition 6. Let φ be a formula on U, the operators ■ or ◆ occur in φ, and $(P^n, \leqslant)$ be the structure on U^n. For $(U^n, R) \in P^n$, the notation $(U^n, R) \models \varphi$ is used to denote φ is roughly true in (U^n, R), which is recursively defined as follows:

1) If $\varphi = \blacksquare\psi$, or $\varphi = \blacklozenge\psi$, then:

i) $(U^n, R) \models \blacksquare\psi$ if and only if for every $(U^n, R_1) \in P^n$, if $(U^n, R) \leqslant (U^n, R_1)$, then $(U^n, R_1) \models \psi$.

ii) $(U^n, R) \models \blacklozenge\psi$ if and only if there exists $(U^n, R_1) \in P^n$ and $(U^n, R) \leqslant (U^n, R_1)$, such that $(U^n, R_1) \models \psi$.

2) If $\varphi = \neg\blacksquare\psi$, or $\varphi = \neg\blacklozenge\psi$, then:

i) $(U^n, R) \models \neg\blacksquare\psi$ if and only if there exists $(U^n, R_1) \in P^n$ and $(U^n, R) \leqslant (U^n, R_1)$, such that $(U^n, R_1) \models \psi$ fails to hold.

ii) $(U^n, R) \models \neg\blacklozenge\psi$ if and only if for every $(U^n, R_1) \in P^n$, if $(U^n, R) \leqslant (U^n, R_1)$, then $(U^n, R_1) \models \psi$ does not hold.

3) If $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \psi_1 \vee \psi_2$, or $\varphi = \psi_1 \rightarrow \psi_2$, where the operators ■ or ◆ occur in ψ_1 or in ψ_2, then:

i) $(U^n, R) \models \psi_1 \wedge \psi_2$ if and only if $(U^n, R) \models \psi_1$ and $(U^n, R) \models \psi_2$.

ii) $(U^n, R) \models \psi_1 \vee \psi_2$ if and only if $(U^n, R) \models \psi_1$ or $(U^n, R) \models \psi_2$.

iii) $(U^n, R) \models \psi_1 \rightarrow \psi_2$ if and only if $(U^n, R) \models \psi_1$ implies $(U^n, R) \models \psi_2$. ▮

This definition shows that the semantics of the operators ■ and ◆ is similar to the interpretation of the modal operators □ and ◇ which express "necessity" and "possibility" respectively in modal logic. Thus, we might think that ■ expresses "rough necessity" and ◆

expresses “rough possibility”.

From 1) and 2) of definition 6, we know that $(U^n, R)\models \varphi$ is closely connected with $(U^n, R_1)\models \psi$. When ψ does not involve the operators ■ and ◆(of course, ψ may be an atomic formula), $(U^n, R_1)\models \psi$ represents that ψ is roughly true in (U^n, R_1)(see definition 5). Thus, $(U^n, R)\models \varphi$ has close links with rough truth. Also, from 3) of definition 6, we can see that $(U^n, R)\models \varphi$ is defined by $(U^n, R)\models \psi_1$ or $(U^n, R)\models \psi_2$. Since $(U^n, R)\models \psi_1$ and $(U^n, R)\models \psi_2$ are eventually returned to consider the items 1) or 2) of definition 6, $(U^n, R)\models \varphi$ also correlates with rough truth. Thus, $(U^n, R)\models \varphi$ is used to denote rough truth of φ in (U^n, R) (see definition 6).

The above discussion is mainly aimed at the definition of rough truth for the formulas on U. From definitions 5 and 6, we know that rough truth of formula φ falls into two cases:

a) The operators ■ and ◆ do not appear in φ. In this case, rough truth of φ is defined in an approximation space (U^n, R) which is analogous to a semantic model in classical predicate logic [7]. In fact, a semantic mode in classical predicate logic is generally denoted by (D, τ), where D is a non-empty set, τ is a function, and every classical formula can be interpreted as a statement which is true or false in (D, τ). The approximation space $M=(U^n, R)$ and the model (D, τ) play a similar role.

b) The operators ■ or ◆ occur in φ. In this case, we take $(P^n, \leqslant)$ as a structure to define rough truth of φ, which follows the way in modal logic. In fact, The formulas $\Box\psi$ and $\Diamond\psi$ in modal logic is interpreted in the structure $(W, \leqslant, V)$ called a Kripke model [1][2], where W is a set of states, $\leqslant$ is a binary relation on W, and V is a function. $\Box\psi$ is true at state $w(\in W)$ if for every $w'\in W$, when $w\leqslant w'$, ψ is true at state w'. $\Diamond\psi$ is true at state w if there exists $w'\in W$ and $w\leqslant w'$, such that ψ is true at state w'. The definition about $(U^n, R)\models$ ■ψ or $(U^n, R)\models$ ◆ψ just imitate the definition in model logic. Every approximation space (U^n, R) in $(P^n, \leqslant)$ corresponds to a state in the Kripke model $(W, \leqslant, V)$. Thus the structure $(P^n, \leqslant)$ can be regarded as a particular Kripke model. The particularity is obvious because P^n consists of approximation spaces.

From definitions 5 and 6, we know that the logical values of the formula φ are relevant to the semantic set $|\varphi|$, which is different from the definition of logical values in modal logic, or in classical logic. The semantic set will lead to the concept of granules.

III. Granules and Granular Computing

In recent years, many scholars focus their attention on the study of granular computing which is an active research topic in information science. From the informal point of view, scholars generally regard granular computing as various combinations or computations of granules, such as the union of granules, the intersection of granules, upper approximation about granules, etc. Thus, granular computing is based on granules. However, what is a granule? As an informal explanation, a granule is viewed as a part of a whole, or is a clump of elements drawn together from the whole. In order to make data processing, we often need to divide a whole into parts which can be viewed as clumps of elements drawn from the whole. Generally, the method of getting a clump is based on a property that the elements of the clump satisfy. Since a formula not involving the operators ■ and ◆ actually describes a property, we will use the formulas to produce granules. Let φ be an n-place formula, and the operators ■ and ◆ do not occur in φ. By definition 4 we can get the semantic set $|\varphi|$, and $|\varphi|\subseteq U^n$ $(n\geqslant 1)$. If U^n is taken as a whole, then $|\varphi|$ is a part of the whole U^n, which gives rise to a definition about granules.

Definition 7. Let φ be an n-place formula on U, and φ do not involve the operators ■ and ◆. The semantic set $|\varphi|$ is called a granule corresponding to φ. ▌

It follows from definition 4 that $|\varphi|$ is a subset of U^n, i.e. $|\varphi|\subseteq U^n$. Moreover, if R is an equivalence relation on U^n, we have $R^*(|\varphi|)\subseteq U^n$. The discussion in [8] shows that there exist a formulas φ_1 on U, such that $|\varphi_1|=R^*(|\varphi|)$. So, R-upper approximation $R^*(|\varphi|)$ is a granule corresponding to φ_1. Thus, the logical values of φ, given in definition 5, are determined by different cases of granules.

As mentioned above, the informal understanding about granular computing can be regarded as combinations or computations of granules. In [8], the authors have made a definition of granular computing, which shows that granular computing is various correspondences from G^n to G, where G is a set of granules, and $n\geqslant 1$. Specifically, the correspondences from $|\varphi|$ to $\sim|\varphi|$ $(=|\neg\varphi|)$, from $|\varphi|$ and $|\psi|$ to $|\varphi|\cap|\psi|(=|\varphi\wedge\psi|)$, from $|\varphi|$ and $|\psi|$ to $|\varphi|\cup|\psi|(=|\varphi\vee\psi|)$, from $|\varphi|$ and $|\psi|$ to $|\neg\varphi|\cup|\psi|$ $(=|\varphi\rightarrow\psi|)$, and from $|\varphi|$ to $R^*(|\varphi|)$ are granular computing. We are not going to define what is granular computing in this paper. For the detailed discussion, we refer the reader to ref. [8]. As long as we remember that these correspondences are granular computing, it is enough for us to make the following discussion.

IV. Rough Truth of Rough Axioms

We will introduce rough axioms in this section, and analyze whether the rough axioms are roughly true in each $(U^n, R)\in P^n$. The structure $(P^n, \leqslant)$ will be taken as a model in following investigation. Firstly, we discuss the properties of the structure $(P^n, \leqslant)$.

A. Properties of $(P^n, \leqslant)$

Let U be a universal set, and $(P^n, \leqslant)$ be the structure on U^n. Since U^n varies with the change of U or n, U and n have close links with P^n, as well as with the structure $(P^n, \leqslant)$.

In order to make research, it is necessary to discuss the properties of $(P^n, \leqslant)$. Some properties are summarized as the following propositions.

Proposition 1. Let φ be an n-place formula on U, and φ do not involve the operators ■ and ◆. Let $(P^n, \leqslant)$ be the structure on U^n. For (U^n, R_1), $(U^n, R_2)\in P^n$, if $(U^n, R_1) \leqslant (U^n, R_2)$, then $(U^n, R_1)\models \varphi$ implies $(U^n, R_2)\models \varphi$.

Proof For simplicity, an element $<t_1,\ldots, t_n>\in U^n$ is denoted by $\boldsymbol{x}$ or $\boldsymbol{b}$, i.e. $\boldsymbol{x}=<t_1,\ldots, t_n>$ or $\boldsymbol{b}=<t_1,\ldots, t_n>$.

Suppose that $(U^n, R_1)\models \varphi$. By $(U^n, R_1)\leqslant(U^n, R_2)$, we have $R_1\subseteq R_2$. Let $U^n/R_1=\{S_1,\ldots, S_r\}$ and $U^n/R_2=\{T_1,\ldots, T_s\}$ be the partitions of U^n relative to R_1 and R_2 respectively. Of course, $U^n=S_1\cup\ldots\cup S_r=T_1\cup\ldots\cup T_s$. For any $T_i\in U^n/R_2$, there is an element $\boldsymbol{b}\in U^n$, such that $T_i=\{\boldsymbol{x}\mid \boldsymbol{x}\in U^n \text{ and } <\boldsymbol{b}, \boldsymbol{x}>\in R_2\}$. Let $S_j=\{\boldsymbol{x}\mid \boldsymbol{x}\in U^n \text{ and } <\boldsymbol{b}, \boldsymbol{x}>\in R_1\}$, then $S_j\in U^n/R_1$. From $R_1\subseteq R_2$ we know $S_j\subseteq T_i$. It follows from $(U^n, R_1)\models\varphi$ that $R_1{}^*(|\varphi|)=U^n$ which means $S_j\cap|\varphi|\neq\varnothing$. By $S_j\subseteq T_i$, we get $T_i\cap|\varphi|\neq\varnothing$. Thus for any $T_i\in U^n/R_2$, $T_i\cap|\varphi|\neq\varnothing$ from which we derive $R_2{}^*(|\varphi|)=U^n$. Hence, $(U^n, R_2)\models\varphi$. ▌

We know that for a partially ordered set $(L, \leqq)$, if any two elements a, $b(\in L)$ have a least upper bound $a\vee b(\in L)$, and a greatest lower bound $a\wedge b(\in L)$ about the partially ordered relation $\leqq$, then $(L, \leqq)$ is called a lattice [9]. Consider the partially ordered set $(R^n, \subseteq)$, where $R^n=\{R\mid R \text{ is an equivalence relation on } U^n\}$. We have the following proposition.

Proposition 2. $(R^n, \subseteq)$ is a lattice.

Proof For R_1, $R_2\in R^n$, $R_1\cap R_2$ is also an equivalence relation on U^n. Thus $R_1\cap R_2\in R^n$, and $R_1\cap R_2$ is the greatest lower bound of R_1 and R_2 about the relation $\subseteq$. On the other hand, suppose that $t(R_1\cup R_2)$ is the transitive closure of $R_1\cup R_2$, then $t(R_1\cup R_2)$ is the least equivalence relation containing R_1 and R_2(see P155 in [9]). Hence $t(R_1\cup R_2)\in R^n$, and $t(R_1\cup R_2)$ is the least upper bound of R_1 and R_2 about $\subseteq$. Thus, $(R^n, \subseteq)$ is a lattice. ▌

Because P^n is closely connected with R^n, and for (U^n, R_1), $(U^n, R_2)\in P^n$, $(U^n, R_1)\leqslant(U^n, R_2)$ if and only if $R_1\subseteq R_2$. By proposition 2, we get the following result:

Proposition 3. $(P^n, \leqslant)$ is a lattice. ▌

Thus, for any elements (U^n, R_1), $(U^n, R_2)\in P^n$, there is an element $(U^n, R_3)\in P^n$, such that (U^n, R_3) is the least upper bound of (U^n, R_1) and (U^n, R_2); meanwhile, there is an element $(U^n, R_4)\in P^n$, and (U^n, R_4) is the greatest lower bound of (U^n, R_1) and (U^n, R_2).

The conclusions in propositions 1 and 3 will be used in the following proofs.

B. Rough Axioms, Rough Rules and Rough Validity

In modal logic, the formal system $\boldsymbol{S}_5$ consists of the following axioms which involve the modal operators $\Box$ or $\Diamond$:

① $\Box\varphi\to\varphi$,

② $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$,

③ $\Box\varphi\to\Box\Box\varphi$,

④ $\Diamond\varphi\to\Box\Diamond\varphi$.

Generally, the formal system consisting of the axioms ① and ② is called the system T, and the formal system consisting of the axioms ①, ② and ③ is called the system $\boldsymbol{S}_4$. Based on the axioms, the formal deduction in T, $\boldsymbol{S}_4$ or $\boldsymbol{S}_5$ can be implemented by the following rules of deduction:

⑤ $\varphi\to\psi, \varphi\Longrightarrow\psi$ (i.e. ψ is a direct consequence of $\varphi\to\psi$ and φ).

⑥ $\varphi\Longrightarrow\Box\varphi$ (i.e. $\Box\varphi$ is a direct consequence of φ when φ is an axiom).

Axioms ①—④ are formulas of modal logic. When they are interpreted in a Kripke model, their logical values take truth or falsity. Hence, modal logic is two-valued logic. It is the same as classical logic. Also, for rules ⑤ and ⑥, we always concern ourselves with the property that the rules keep truth. For example, for rule ⑤ $\varphi\to\psi, \varphi\Longrightarrow\psi$, we are concerned about what truth value, truth or falsity, taken by $\varphi\to\psi$ and φ can keep ψ being true. And for rule ⑥ $\varphi\Longrightarrow\Box\varphi$, we generally pay our attention to the condition of φ which guarantees $\Box\varphi$ is true. These also show that modal logic only correlates with truth and falsity which are taken as logical values of formulas. In order to make different research, our investigation on ①—⑥ will focus on rough truth. To this end, we replace the operators $\Box$ and $\Diamond$ in ①—⑥ by $\blacksquare$ and $\blacklozenge$ respectively, and get the following expressions:

(1) $\blacksquare\varphi\to\varphi$,

(2) $\blacksquare(\varphi\to\psi)\to(\blacksquare\varphi\to\blacksquare\psi)$,

(3) $\blacksquare\varphi\to\blacksquare\blacksquare\varphi$,

(4) $\blacklozenge\varphi\to\blacksquare\blacklozenge\varphi$,

(5) $\varphi\to\psi, \varphi\Longrightarrow\psi$,

(6) $\varphi\Longrightarrow\blacksquare\varphi$.

We call (1)—(4) rough axioms, (5) and (6) rough rules. It is clear that rough axioms (1)—(4) are formulas of definition 3.

Let us consider the formula $\varphi\to\psi$ in which the connective $\to$ occurs. In classical logic, when truth of φ implies truth of ψ, the formula $\varphi\to\psi$ is said to be valid. Proceeding in a manner similar to this concept, we introduce rough validity.

Definition 8. Let $(P^n, \leqslant)$ be the structure on U^n, and $\varphi\to\psi$ be an n-place formula on U. The formula $\varphi\to\psi$ is said to be roughly valid in $(P^n, \leqslant)$, if for every $(U^n, R)\in P^n$, rough truth of φ in (U^n, R) implies rough truth of ψ in (U^n, R), i.e. $(U^n, R)\models\varphi$ implies $(U^n, R)\models\psi$. ▌

Rough validity of $\varphi\to\psi$ in $(P^n, \leqslant)$ means that for every $(U^n, R)\in P^n$, the formula $\varphi\to\psi$ is roughly true in (U^n, R), i.e. $(U^n, R)\models\varphi\to\psi$(see definition 6), where the operators $\blacksquare$ or $\blacklozenge$ appear in φ or in ψ. So, research on rough validity of $\varphi\to\psi$ is to investigate whether rough truth of the antecedent φ can imply rough truth of the consequent ψ, or to investigate whether $\varphi\to\psi$ is roughly true in each $(U^n, R)\in P^n$

C. Rough Validity of Rough Axioms

We now discuss rough validity of rough axioms (1)—(4). The above analysis indicates that for a rough axiom, such as (4) $\blacklozenge\varphi\to\blacksquare\blacklozenge\varphi$, rough validity of it in $(P^n, \leqslant)$ is to decide whether rough truth of the antecedent $\blacklozenge\varphi$

can imply rough truth of the consequent ■◆φ in each $(U^n, R)\in P^n$, i.e. whether $(U^n, R)\models$ ◆φ can imply $(U^n, R)\models$ ■◆φ for every $(U^n, R)\in P^n$. Now we first investigate this rough axiom.

Theorem 1. Let φ be an n-place formula on U, and φ do not involve the operators ■ and ◆. If $(P^n, \leqslant)$ is the structure on U^n, then the rough axiom (4) ◆φ → ■◆φ is roughly valid in $(P^n, \leqslant)$.

Proof For $(U^n, R)\in P^n$, suppose that $(U^n, R)\models$ ◆φ. Let $(U^n, R_1)\in P^n$, and $(U^n, R)\leqslant(U^n, R_1)$. Since $(U^n, R)\models$ ◆φ, by definition 6, there exists a $(U^n, R_2)\in P^n$, and $(U^n, R)\leqslant(U^n, R_2)$, such that $(U^n, R_2)\models\varphi$. Consider (U^n, R_1) and (U^n, R_2). By proposition 3, there must be a $(U^n, R_3)\in P^n$, such that (U^n, R_3) is the least upper bound of (U^n, R_1) and (U^n, R_2). Therefore $(U^n, R_1)\leqslant(U^n, R_3)$ and $(U^n, R_2)\leqslant(U^n, R_3)$. Since $(U^n, R_2)\leqslant(U^n, R_3)$, by proposition 1, $(U^n, R_3)\models\varphi$. Since $(U^n, R_1)\leqslant(U^n, R_3)$, we derive $(U^n, R_1)\models$ ◆φ from definition 6. It has been proved that for any $(U^n, R_1)\in P^n$, if $(U^n, R)\leqslant(U^n, R_1)$, then $(U^n, R_1)\models$ ◆φ. By definition 6 again, we have $(U^n, R)\models$ ■◆φ. Thus, rough axiom (4) ◆φ → ■◆φ is roughly valid in $(P^n, \leqslant)$. ▌

Because $(U^n, R_i)\models\varphi$ (i=2, 3) is defined by $R_i^*(|\varphi|)= U^n$, the granules $|\varphi|$, $R_2^*(|\varphi|)$ and $R_3^*(|\varphi|)$ are linked to the proof of this theorem. It has been mentioned in section 3 that the correspondence from $|\varphi|$ to $R_2^*(|\varphi|)$, or to $R_3^*(|\varphi|)$ is granular computing. Thus, the process of judging rough validity of rough axiom (4) ◆φ → ■◆φ is relevant to granular computing. By this way, we can investigate other rough axioms.

Let us examine rough axiom (2) ■$(\varphi\to\psi)\to$ (■$\varphi\to$ ■ψ). Its equivalent form can be expressed as (2) (■$(\varphi\to\psi)\wedge$■φ) → ■ψ. But we must point out that this rough axiom is not roughly valid, i.e. rough truth of ■$(\varphi\to\psi)\wedge$ ■φ may fail to imply rough truth of ■ψ in an approximate space (U^n, R), which can be illustrated by the following example.

Example. Let $U=\{u_1, u_2, u_3\}$, and let $R=U\times U$. It is clear that R is an equivalence relation on U, thus (U, R) is an approximation space which belongs to P^n, where n=1. Also, we can get $U/R=\{U\}$, the partition of U relative to R. Let $H=\{<u_1, u_1>\}$ and $Q=\{<u_2, u_3>\}$. H and Q are binary-place relations on U. Consider the formulas $\varphi=H(x_1, u_1)$ and $\psi=Q(x_1, u_1)$. Because x_1 is a variable and u_1 is a constant, φ and ψ are unary-place formulas on U. Since $|\varphi|=|H(x_1, u_1)|=\{u_1\}$, $|\psi|=|Q(x_1, u_1)|=\varnothing$ and $|\neg\varphi|=\sim|\varphi|=U-|\varphi|=U-\{u_1\}=\{u_2, u_3\}$, we have $R^*(|\varphi|)=U$, $R^*(|\neg\varphi|)=U$ and $R^*(|\psi|)=\varnothing$. Note that (U, R) is the greatest element of the lattice $(P^n, \leqslant)$, where n=1. Thus for $(U^n, R_1)\in P^n$, if $(U, R)\leqslant(U^n, R_1)$, then $(U^n, R_1)=(U, R)$. In this case, making use of the property: $R^*(|\varphi\to\psi|)=R^*(|\neg\varphi|\cup|\psi|)=R^*(|\neg\varphi|\cup\varnothing)=R^*(|\neg\varphi|)=U$, and by $R^*(|\varphi|)=U$, we have $(U^n, R)\models$ ■$(\varphi\to\psi)$ and $(U^n, R)\models$ ■φ. So, $(U^n, R)\models$ ■$(\varphi\to\psi)\wedge$■φ. Whereas, it follows from $R^*(|\psi|)=\varnothing$ that $(U^n, R)\models$ ■ψ fails to hold. Thus the rough axiom (2) (■$(\varphi\to\psi)\wedge$■φ) → ■ψ is not roughly valid in $(P^n, \leqslant)$, where n=1. ▌

However, if we slightly change the form of rough axiom (2) (■$(\varphi\to\psi)\wedge$■φ) → ■ψ, rough validity will hold. The following paragraph explains how we change the rough axiom:

Suppose that φ is a formula that does not involve ■ and ◆. For $(U^n, R)\in P^n$, if φ is true in (U^n, R), i.e. $|\varphi|=U^n$(see definition 5), then the granule $|\varphi|$ does not depend on the equivalence relation R, it follows that for any $(U^n, R_1)\in P^n$, φ is true in (U^n, R_1). Thus, we introduce the notation $\Box\varphi$ which can be regarded as a special formula, although the formulas in definition 3 are not relevant to the symbol $\Box$. Also, we use $(U^n, R)\models\Box\varphi$ to express $U^n\models\varphi$ or $|\varphi|=U^n$(see definition 5), which represents that φ is independent of R, and for each $(U^n, R_1)\in P^n$, φ is true in (U^n, R_1), i.e. $U^n\models\varphi$ or $|\varphi|=U^n$. Now, consider the formula (2)′ (■$(\varphi\to\psi)\wedge\Box\varphi$) → ■$\psi$. It is clear that (2)′ is a new version of rough axiom (2) (■$(\varphi\to\psi)\wedge$■φ) → ■ψ. We also refer to (2)′ (■$(\varphi\to\psi)\wedge\Box\varphi$) → ■$\psi$ as a rough axiom. The next theorem shows that (2)′ (■$(\varphi\to\psi)\wedge\Box\varphi$) → ■$\psi$ is roughly valid in $(P^n, \leqslant)$.

Theorem 2. Let φ and ψ be n-place formulas on U, and do not involve the operators ■ and ◆. If $(P^n, \leqslant)$ is the structure on U^n, then rough axiom (2)′ (■$(\varphi\to\psi)\wedge\Box\varphi$) → ■$\psi$ is roughly valid in $(P^n, \leqslant)$.

Proof For $(U^n, R)\in P^n$, suppose that $(U^n, R)\models$ ■$(\varphi\to\psi)\wedge\Box\varphi$. By definition 6, $(U^n, R)\models$ ■$(\varphi\to\psi)$ and $(U^n, R)\models\Box\varphi$. From $(U^n, R)\models$ ■$(\varphi\to\psi)$, we get $(U^n, R_1)\models(\varphi\to\psi)$ for $(U^n, R_1)\in P^n$ and $(U^n, R)\leqslant(U^n, R_1)$. Since φ and ψ do not involve the operators ■ and ◆, It follows from definition 5 that $R_1^*(|\varphi\to\psi|)=U^n$. Meanwhile, by $(U^n, R)\models\Box\varphi$, we have $|\varphi|=U^n$. Hence, $R_1^*(|\varphi\to\psi|)=R_1^*(|\neg\varphi|\cup|\psi|)=R_1^*((\sim|\varphi|)\cup|\psi|)=R_1^*((\sim U^n)\cup|\psi|)=R_1^*(\varnothing\cup|\psi|)=R_1^*(|\psi|)$. So, $R_1^*(|\psi|)=U^n$, that is $(U^n, R_1)\models\psi$. Hence, It has been proved that for any $(U^n, R_1)\in P^n$, if $(U^n, R)\leqslant(U^n, R_1)$, then $(U^n, R_1)\models\psi$. By definition 6, $(U^n, R)\models$ ■ψ. Thus, rough axiom (2)′ (■$(\varphi\to\psi)\wedge\Box\varphi$) → ■$\psi$ is roughly valid in $(P^n, \leqslant)$. ▌

Since the relation $\leqslant$ is reflective and transitive on P^n, we are able to prove rough axioms (1) and (3) are roughly valid in $(P^n, \leqslant)$.

Theorem 3. Let φ be an n-place formula on U, and φ do not involve the operators ■ and ◆. If $(P^n, \leqslant)$ is the structure on U^n, then:

1) Rough axiom (1) ■$\varphi\to\varphi$ is roughly valid in $(P^n, \leqslant)$.

2) Rough axiom (3) ■φ → ■■φ is roughly valid in $(P^n, \leqslant)$.

Proof 1) For $(U^n, R)\in P^n$, suppose that $(U^n, R)\models$ ■φ. By definition 6, we have $(U^n, R_1)\models\varphi$ for any $(U^n, R_1)\in P^n$ and $(U^n, R)\leqslant(U^n, R_1)$. Since the relation $\leqslant$ is reflective, we have $(U^n, R)\leqslant(U^n, R)$. Thus $(U^n, R)\models\varphi$. Therefore, rough axiom (1) ■$\varphi\to\varphi$ is roughly valid in

$(P^n, \leqslant)$.

2) For $(U^n, R) \in P^n$, suppose that $(U^n, R) \models ■\varphi$. Let $(U^n, R_1) \in P^n$ and $(U^n, R) \leqslant (U^n, R_1)$. For any $(U^n, R_2) \in P^n$ and $(U^n, R_1) \leqslant (U^n, R_2)$, since $\leqslant$ is transitive, we have $(U^n, R) \leqslant (U^n, R_2)$. It follows from $(U^n, R) \models ■\varphi$ that $(U^n, R_2) \models \varphi$. This indicates that for any $(U^n, R_2) \in P^n$, when $(U^n, R_1) \leqslant (U^n, R_2)$, $(U^n, R_2) \models \varphi$. By definition 6, $(U^n, R_1) \models ■\varphi$. Thus, we have proved the result that for any $(U^n, R_1) \in P^n$, when $(U^n, R) \leqslant (U^n, R_1)$, $(U^n, R_1) \models ■\varphi$. By use of definition 6 again, $(U^n, R) \models ■■\varphi$. Hence, rough axiom (3) $■\varphi \to ■■\varphi$ is roughly valid in $(P^n, \leqslant)$. ▌

Note that the condition "φ does not involve the operators ■ and ◆" can be removed from this theorem.

There are other formal systems in modal logic which include the following axioms:

⑦ $\Box\varphi \to \Diamond\varphi$,

⑧ $\varphi \to \Box\Diamond\varphi$.

Corresponding to ⑦ and ⑧, we get the following rough axioms (7) and (8).

(7) $■\varphi \to ◆\varphi$,

(8) $\varphi \to ■◆\varphi$.

It is easy to prove that (7) $■\varphi \to ◆\varphi$ and (8) $\varphi \to ■◆\varphi$ are roughly valid in $(P^n, \leqslant)$. In rough axiom (8), formula φ should not involve the operators ■ and ◆.

From definition 5, we know that rough truth of φ is linked to the R-upper approximation $R^*(|\varphi|)$ that is a granule, i.e. there is a formula ψ, such that $|\psi|=R^*(|\varphi|)$[8]. At the same time, since rough validity of the rough axioms has close links with rough truth(see definition 8), rough validity is relevant to granules. Also notice that the correspondence from $|\varphi|$ to $R^*(|\varphi|)$ is granular computing [8]. The processes of determining whether rough axioms (1), (2)', (3) and (4) are roughly valid in $(P^n, \leqslant)$ are supported by granular computing which can be viewed as an approach to data processing.

D. The Condition of Keeping Rough Truth

The role of a rule of deduction is to get a new formula from other formulas. For instance, by rough rule (5) $\varphi \to \psi$, $\varphi \Longrightarrow \psi$, the formula ψ can be derived from the formulas $\varphi \to \psi$ and φ. Also, rough rule (6) $\varphi \Longrightarrow ■\varphi$ is a process of deduction from φ to $■\varphi$. For rough rule (5) $\varphi \to \psi$, $\varphi \Longrightarrow \psi$, what condition that the formulas $\varphi \to \psi$ and φ satisfy can make ψ roughly true. And for rough rule (6) $\varphi \Longrightarrow ■\varphi$, what condition of φ can guarantee $■\varphi$ is roughly true. The following theorems will give the answers to these questions.

Theorem 4. Let φ and ψ be n-place formulas on U, and the operators ■ or ◆ occur in φ or in ψ. Let $(P^n, \leqslant)$ be the structure on U^n. For $(U^n, R) \in P^n$, if $\varphi \to \psi$ and φ are roughly true in (U^n, R), then ψ is also roughly true in (U^n, R).

Proof By definition 6, if $\varphi \to \psi$ is roughly true in (U^n, R) (i.e. $(U^n, R) \models \varphi \to \psi$), then rough truth of φ (i.e. $(U^n, R) \models \varphi$) implies rough truth of ψ (i.e. $(U^n, R) \models \psi$). Now, suppose that $\varphi \to \psi$ and φ are roughly true in (U^n, R). Then it follows from definition 6 that ψ is roughly true in (U^n, R). ▌

So, as long as $\varphi \to \psi$ and φ are roughly true in (U^n, R), rough rule (5) $\varphi \to \psi$, $\varphi \Longrightarrow \psi$ will keep rough truth, where the operators ■ or ◆ occur in φ or in ψ.

When the operators ■ or ◆ occur in φ or in ψ, and $\varphi \to \psi$ is roughly valid in $(P^n, \leqslant)$, by definition 8 we know that if φ is roughly true, then ψ is also roughly true in each $(U^n, R) \in P^n$, or $\varphi \to \psi$ is roughly true in (U^n, R) (see definition 6). Thus, when $\varphi \to \psi$ is roughly valid in $(P^n, \leqslant)$, rough truth of φ can guarantee rough truth of ψ in each $(U^n, R) \in P^n$. Since the conclusions in theorems 1－3 show that rough axioms (1), (2)', (3) and (4) are roughly valid in $(P^n, \leqslant)$, for each of the rough axioms, such as (3) $■\varphi \to ■■\varphi$, rough truth of the antecedent $■\varphi$ can imply rough truth of the consequent $■■\varphi$ in each $(U^n, R) \in P^n$. Thus, if a rough axiom, such as (3) $■\varphi \to ■■\varphi$, is taken as the formula $\varphi \to \psi$ in rough rule (5) $\varphi \to \psi$, $\varphi \Longrightarrow \psi$, then as long as the antecedent $■\varphi$ is roughly true in (U^n, R), the consequent $■■\varphi$ deduced by this rule is certainly roughly true in (U^n, R).

Theorem 5. Let φ be an n-place formula on U, and φ do not involve the operators ■ and ◆. Let $(P^n, \leqslant)$ be the structure on U^n. For each $(U^n, R) \in P^n$, if $(U^n, R) \models \varphi$, then $(U^n, R) \models ■\varphi$.

Proof Suppose that $(U^n, R) \models \varphi$. Since the operators ■ and ◆ do not occur in the formula φ, it follows from proposition 1 that we have $(U^n, R_1) \models \varphi$ for any $(U^n, R_1) \in P^n$ and $(U^n, R) \leqslant (U^n, R_1)$. Thus, as long as $(U^n, R_1) \in P^n$ and $(U^n, R) \leqslant (U^n, R_1)$, $(U^n, R_1) \models \varphi$ is true. From definition 6, we conclude $(U^n, R) \models ■\varphi$. ▌

This theorem illustrates that for $(U^n, R) \in P^n$, if φ is roughly true in (U^n, R), then $■\varphi$ is certainly roughly true in (U^n, R), where φ does not involve the operators ■ and ◆. In this case, as long as φ is roughly true, rough rule (6) $\varphi \Longrightarrow ■\varphi$ will keep rough truth.

In theorems 1－3 the operators ■ and ◆ do not occur in the formulas φ and ψ, and theorem 5 requires the formula φ not to involve ■ and ◆. If these conditions are removed, and ■ or ◆ may occur in φ or in ψ, could we still get these theorems? This will be a problem we are going to investigate in the future.

Making use of the formulas on a universal set U, and based on $(P^n, \leqslant)$, the structure on U^n, we have made researches into rough validity of the rough axioms which have the same forms as the axioms in modal logic. Rough validity of a rough axiom is to investigate whether rough truth of its antecedent can imply rough truth of its consequent in each $(U^n, R) \in P^n$. So, rough validity has close links with rough truth. The connection between rough validity and the rough axioms is the major study in this paper, which reflects the idea of integrating rough logic with modal logic. Also, the method of combining logic with granular computing plays an important role in our proofs. Although our research mainly focuses on theoretical aspect, the study has laid a foundation for applications, and we will take up the research in the future.

V. Conclusion

The study about rough validity of the rough axioms is the important work in this paper. Because rough truth originates in rough logic, and the rough axioms have the same forms as the axioms of modal logic, our study embodies the idea of integrating rough logic with modal logic. Meanwhile, the study of combining granular computing with mathematical logic forms an approach to data processing.

From the definition we know that rough validity is closely related to rough truth which is a logical value. In classical logic, the deduction that depends on logical values is referred to as semantic deduction. Observe the discussion in this paper. The analysis on rough validity of rough axioms (1)—(4) has close links with the deduction produced by rough rules (5) or (6). Because rough validity is related to rough truth that is a logical value, the process of deciding whether the rough axioms are roughly valid is therefore semantic deduction. We might refer to it as roughly semantic deduction.

Now, consider rough axioms (1), (2)′, (3), (4), as well as rough rules (5) and (6) which form a formal system. Based on the system, it is possible to conduct formal deduction. The study in this paper shows that corresponding to the formal deduction, the roughly semantic deduction keeps rough truth. In this case, the formal system is said to have the property of soundness. But in order to conduct formal deduction in the formal system, and make the soundness true, it is necessary to analyze rough validity of every axiom in classical logic, because every formal system in modal logic is an extension of a formal system in classical logic. However the above analysis on rough validity of the rough axioms has laid a foundation for such efforts. Further research is required. Importantly, in [10], [11] and [12] the authors have introduced a method correlating with roughly semantic deduction which may be used in further investigation. Lastly, we point out that granular computing plays a key role in the proofs of the theorems.

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