Duality of Multi-objective Programming

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***Abseact*—In this paper, a class of multi-objective programming is considered, in which related functions are- $B-(p,r,a)$-invex functions, Mond-Weir dual problem is researched, many duality theorems are proved under weaker convexity.**

***Index Term*— $B-(p,r,a)$-invex function， multi-objective programming，duality**

I. INTRODUCTION

The convexity theory plays an important role in many aspects of mathematical programming. In recent years, in order to relax convexity assumption, various generalized convexity notions have been obtained. One of them is the concept of $B-(p,r)$invexity defined by T.Antczak [1], which extended the class of $B-$ invex functions with respect to η and b and the classes of (p,r)invex functions with respect to η [2][3]. He proved some necessary and sufficient conditions for- $B-(p,r)$ invexity and showed the relationships between the defined $B-(p,r)$-invex functions and other classes of invex functions. Later Antczak defined a classes of generalized invex functions[4], that is $B-(p,r)$ pseudo-invex functions, strictly $B-(p,r)$ pseudo-invex functions, and $B-(p,r)$ quasi-invex functions, considered single objective mathe -matical programming problem involving $B-(p,r)$ pseudo-invex functions, $B-(p,r)$ quasi-invex functions and obtained some sufficient optimality conditions. Qing xing Zhang[5][6]defined B -arcwise connected functions, $(v,\rho)_{h,\varphi}$-type I functions, studied multiobjective progra -mming problem in which involving functions belong to the introduced classes of functions, Xiangyou Li[7] discussed saddle-point conditions for multi-objective fractional programming.

Under different assumption of convexity, several auth -ors establish various duality results. Zhang Ying, Zhu Bo and Xu yingtao discussed nonsmooth programming by a class of Lipschitz $B-(p,r)$ invex function, studied Mond-Weir type dual and Wolfe type dual, derived many dual conditions, Liang Zhi'an, Zhang Zhenhua [9] considered duality for uniform invex multi-objective programming, derived many dual conditions, MorganA. Hanson, Rita Pini and Chanchal Singh [10] researched multiobjective programming problem, using Lagrange multiplier conditions, established many sufficiency results, proved weak, strong and converse duality theorems in the Mond-Weir setting by V-type I-invex functions, Mohamed Hachimi, Brahim Aghezzaf [11] introduced generalized (F,ρ,α,d) type I functions, researched differentiable multi-objective programming, obtained several sufficient optimality conditions, proved weak and strong duality theorems for mixed type duality.

In this paper, we introduce new classes of generalized invex function, classes of $B-(p,r,a)$-invex functions, $B-(p,r,a)$quasi-invex functions, $B-(p,r,a)$pseudo-invex functions and strictly $B-(p,r,a)$ pesudo-invex functions. In this way, we extend $B-(p,r)$ -invex functions, $B-(p,r)$ quasi-invex functions, $B-(p,r)$ pesudo-invex functions and strictly $B-(p,r)$ pesudo-invex functions. Then we research multiobjective programming problem in which corresponding functions belong to the introduced classes of functions, obtain many duality conditions under weaker convexity.

II . DEFINITIONS AND EXAMPLES

Throughout this paper, let R^n be the n-dimensional Eu -clidean space and $R^n{}_+$ be its non negative subset, X be a nonempty open subset of R^n. For the benefit of the reader, we recall concept of $B-(p,r)$-invexity introduced by Antczak in [2] and concept of $B-(p,r,a)$- invexity intr -oduced by Xiangyou Li in [7] .

Definition 2.1[2] Let $u \in X$, The differentiable funct-ion $f: X \to R$ is said to be (strictly) $B-(p,r)$-invex fun -ction with respect to η and b at u on X if there exist fun -ctions $\eta: X\times X \to R^n$, $b: X\times X \to R_+$, $0 \le b(,.,) \le 1$, for all $x \in X$, the inequality

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1) \ge \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-I),$$

$$\mathbf{(> } \mathit{ifx} \neq u\mathbf{),}\ \mathit{for}\mathbf{(}\, p \neq 0, r \neq 0\mathbf{),}$$

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$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1)\geq\nabla f(u)\eta(x,u),$$
$$(>if x\neq u),\ for(p=0,r\neq 0),$$
$$b(x,u)(f(x)-f(u))\geq\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-I),$$
$$(>if x\neq u),\ for(p\neq 0,r=0),$$
$$b(x,u)(f(x)-f(u))\geq\nabla f(u)\eta(x,u),$$
$$(>if x\neq u),\ for(p=0,r=0),$$
holds.

Now, we introduce a definition $B-(p,r,a)$-invex func -tion with respect to η and b at u .

Definition 2.2[7] Let $X\subset R^n$ is a nonempty open set, $u\in X$, the differentiable function $f:X\to R$ is said to be (strictly) $B-(p,r,a)$-invex function with respect to η and b at u if there exist functions $\eta: X\times X\to R^n$, $b: X\times X\to R_+$, $0\le b(,.)\le 1$, $a: X\times X\to R$ for all $x\in X$, the inequality

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1)\geq\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-I)$$
$$+a(x,u),(>if x\neq u),\ for(p\neq 0,r\neq 0),$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1)\geq\nabla f(u)\eta(x,u)$$
$$+a(x,u),(>if x\neq u),\ for(p=0,r\neq 0),$$
$$b(x,u)(f(x)-f(u))\geq\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-I)$$
$$+a(x,u),(>if x\neq u),\ for(p\neq 0,r=0),$$
$$b(x,u)(f(x)-f(u))\geq\nabla f(u)\eta(x,u)$$
$$+a(x,u),(>if x\neq u),\ for(p=0,r=0),$$
holds.

Function $f:X\to R$ is said to be $B-(p,r,a)$-invex function with respect to η and b on X if it is $B-(p,r,a)$ -invex function with respect to the same η and b at each u on X .

Now, we introduce a definition $B-(p,r,a)$ quasi-inv -ex function with respect to η and b at u .

Definition 2.3[7] Let $X\subset R^n$ is a nonempty open set, $u\in X$, the differentiable function $f:X\to R$ is said to be $B-(p,r,a)$ quasi-invex function with respect to η and b at u if there exist functions $\eta: X\times X\to R^n$, $b: X\times X\to R_+$, $0\le b(,.)\le 1$, $a: X\times X\to R$ for all $x\in X$, the inequality

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1)\leq 0\Rightarrow\frac{1}{p}\nabla f(u)$$
$$(e^{p\eta(x,u)}-I)+a(x,u)\leq 0,\ for(p\neq 0,r\neq 0),$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1)\leq 0\Rightarrow\nabla f(u)\eta(x,u)$$
$$+a(x,u)\leq 0,\ for(p=0,r\neq 0),$$
$$b(x,u)(f(x)-f(u))\leq 0\Rightarrow\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-I)$$
$$+a(x,u)\leq 0,\ for(p\neq 0,r=0),$$
$$b(x,u)(f(x)-f(u))\leq 0\Rightarrow\nabla f(u)\eta(x,u)$$
$$+a(x,u)\leq 0,\ for(p=0,r=0),$$
holds.

Function $f:X\to R$ is said to be $B-(p,r,a)$ quasi-invex function with respect to η and b on X if it is $B-(p,r,a)$ quasi-invex function with respect to the same η and b at each u on X .

Now, we introduce a definition $B-(p,r,a)$ pseudo-invex function with respect to η and b at u .

Definition 2.4 [7] Let $X\subset R^n$ is a nonempty open set, $u\in X$, the differentiable function $f:X\to R$ is said to be $B-(p,r,a)$ pseudo-invex function with resp -ect to η and b at u if there exist functions $\eta: X\times X\to R^n$, $b: X\times X\to R_+$, $0\le b(,.)\le 1$, $a: X\times X\to R$, for all $x\in X$, the inequality

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-I)+a(x,u)\geq 0\Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1)\geq 0,\ for(p\neq 0,r\neq 0),$$
$$\nabla f(u)\eta(x,u)+a(x,u)\geq 0\Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1)\geq 0,\ for(p=0,r\neq 0),$$
$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-I)+a(x,u)\geq 0\Rightarrow$$
$$b(x,u)(f(x)-f(u))\geq 0,\ for(p\neq 0,r=0),$$
$$\nabla f(u)\eta(x,u)+a(x,u)\geq 0\Rightarrow$$
$$b(x,u)(f(x)-f(u))\geq 0,\ for(p=0,r=0),$$
holds.

Function $f:X\to R$ is said to be $B-(p,r,a)$ pesudo-invex function with respect to η and b on X if it is $B-(p,r,a)$ pesudo-invex function with respect to the sa -me η and b at each u on X .

Now, we introduce a definition strictly $B-(p,r,a)$ pse -udo-invex function with respect to η and b at u .

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Definition 2.5 [7] Let $X \subset R^n$ is a nonempty open set, $u \in X$, the differentiable function $f: X \to R$ is said to be strictly $B-(p,r,a)$ pseudo-invex function with resp -ect to η and b at u if there exist functions $\eta: X \times X \to R^n$, $b: X \times X \to R_+$, $0 \le b(.,.) \le 1$, $a: X \times X \to R$, for all $x \in X$, the inequality

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - I) + a(x,u) \ge 0 \Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) > 0, \; for(p \ne 0, r \ne 0),$$
$$\nabla f(u)\eta(x,u) + a(x,u) \ge 0 \Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) > 0, \; for(p = 0, r \ne 0),$$
$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - I) + a(x,u) \ge 0 \Rightarrow$$
$$b(x,u)(f(x) - f(u)) > 0, \; for(p \ne 0, r = 0),$$
$$\nabla f(u)\eta(x,u) + a(x,u) \ge 0 \Rightarrow$$
$$b(x,u)(f(x) - f(u)) > 0, \; for(p = 0, r = 0),$$

hold or equivalently have

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) \le 0 \Rightarrow \frac{1}{p}\nabla f(u)$$
$$(e^{p\eta(x,u)} - I) + a(x,u) < 0, \; for(p \ne 0, r \ne 0),$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) \le 0 \Rightarrow \nabla f(u)\eta(x,u)$$
$$+ a(x,u) < 0, \; for(p = 0, r \ne 0),$$
$$b(x,u)(f(x) - f(u)) \le 0 \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - I)$$
$$+ a(x,u) < 0, \; for(p \ne 0, r = 0),$$
$$b(x,u)(f(x) - f(u)) \le 0 \Rightarrow \nabla f(u)\eta(x,u)$$
$$+ a(x,u) < 0, \; for(p = 0, r = 0),$$

hold.

Function $f: X \to R$ is said to be strictly $B-(p,r,a)$ pesudo-invex function with respect to η and b on X if it is $B-(p,r,a)$ pseudo-invex function with respect to the same η and b at each u on X.

Definition 2.6 Let $X \subset R^n$ is a nonempty open set, $u \in X$, the differentiable function $f: X \to R$ is said to be strong $B-(p,r,a)$ pseudo-invex function with respect to η and b at u if there exist functions $\eta: X \times X \to R^n$, $b: X \times X \to R_+$, $0 \le b(.,.) \le 1$, $a: X \times X \to R$, for all $x \in X$, the inequality

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - I) + a(x,u) > 0 \Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) > 0, \; for(p \ne 0, r \ne 0),$$
$$\nabla f(u)\eta(x,u) + a(x,u) > 0 \Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) > 0, \; for(p = 0, r \ne 0),$$
$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - I) + a(x,u) > 0 \Rightarrow$$
$$b(x,u)(f(x) - f(u)) > 0, \; for(p \ne 0, r = 0),$$
$$\nabla f(u)\eta(x,u) + a(x,u) > 0 \Rightarrow$$
$$b(x,u)(f(x) - f(u)) > 0, \; for(p = 0, r = 0),$$

hold.

Function $f: X \to R$ is said to be strong $B-(p,r,a)$ pesudo-invex function with respect to η and b on X if it is $B-(p,r,a)$ pesudo-invex function with respect to the same η and b at each u on X.

Definition 2.7 Let $X \subset R^n$ is a nonempty open set, $u \in X$, the differentiable function $f: X \to R$ is said to be weak $B-(p,r,a)$ pseudo-invex function with respect to η and b at u if there exist functions $\eta: X \times X \to R^n$, $b: X \times X \to R_+$, $0 \le b(.,.) \le 1$, $a: X \times X \to R$, for all $x \in X$, the inequality

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - I) + a(x,u) > 0 \Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) \ge 0, \; for(p \ne 0, r \ne 0),$$
$$\nabla f(u)\eta(x,u) + a(x,u) > 0 \Rightarrow$$
$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) \ge 0, \; for(p = 0, r \ne 0),$$
$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - I) + a(x,u) > 0 \Rightarrow$$
$$b(x,u)(f(x) - f(u)) \ge 0, \; for(p \ne 0, r = 0),$$
$$\nabla f(u)\eta(x,u) + a(x,u) > 0 \Rightarrow$$
$$b(x,u)(f(x) - f(u)) \ge 0, \; for(p = 0, r = 0),$$

hold.

Function $f: X \to R$ is said to be weak $B-(p,r,a)$ pse -udo-invex function with respect to η and b at X if it is weak $B-(p,r,a)$ pesudo-invex function with respect to the same η and b at each u on X.

In above section, $I = (1,\cdots,1) \in R^n$, $e^{(a_1,\cdots,a_n)} = (e^{a_1},\cdots,e^{a_n}) \in R^n$.

When $a(x,u) \ge 0$, $B-(p,r,a)$ -invex function is $B-(p,r)$-invex function, but if $a(x,u) < 0$, $B-(p,r,a)$ invex function may not be $B-(p,r)$-invex function.

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Therefore, adding a parameter $a(x,u)$ means that the $B-(p,r)$ invexity maybe lost.

Now we give several examples about $B-(p,r,a)$-invex function, $B-(p,r,a)$ quasi-invex function, $B-(p,r,a)$ pesudo-invex function with respect to the same η and b.

Example 2.8 We consider a differentiable function $f:R\to R$, defined by $f(x)=\ln(\ln(x^2+e))$, let $\eta(x,u)=-u$, $b(x,u)=\begin{cases}0, x^2<u^2\\ 1, x^2\ge u^2\end{cases}$, then it is not difficult to prove that $f:X\to R$ is $B-(p,r,a)$-invex function with respect to η and b when

$$a(x,u)\le\frac{1}{\ln(u^2+e)}\frac{2u}{(u^2+e)p}(1-e^{-pu}) \text{ for } p\neq 0,$$

(when $a(x,u)\le\dfrac{1}{\ln(u^2+e)}\dfrac{2u^2}{u^2+e}$ for $p=0$).

Example 2.9 We consider a differentiable function $f:R\to R$, defined by $f(x)=\ln(e^{(x^2-1)}+1)$, let $\eta(x,u)=ux^2$, $b(x,u)=\begin{cases}0, x^2<u^2\\ 1, x^2\ge u^2\end{cases}$, then it is not difficult to prove that $f:X\to R$ is $B-(p,r,a)$ pseudo -invex function with respect to η and b when

$$a(x,u)\ge\frac{e^{(u^2-1)}}{e^{(u^2-1)}+1}\frac{2u}{p}(1-e^{-pux^2}) \text{ for } p\neq 0,$$

(when $a(x,u)\ge\dfrac{-2u^2x^2e^{(u^2-1)}}{e^{(u^2-1)}+1}$ for $p=0$).

Example 2.10 We consider a differentiable function, $f:R\to R$, defined by $f(x)=\ln(x^2+1)$, let $\eta(x,u)=-u$, $b(x,u)=\begin{cases}1, x^2<u^2,\\ 0, x^2\ge u^2\end{cases}$, then it is not difficult to prove that $f:X\to R$ is $B-(p,r,a)$ quasi -invex function with respect to η and b when

$$a(x,u)\le\frac{2u}{(u^2+1)p}(1-e^{-pu}) \text{ for } p\neq 0,$$

(when $a(x,u)\le\dfrac{2u^2}{u^2+1}$ for $p=0$).

Now, we give a useful lemma whose simple proof is omitted in the paper.

Lemma 2.11 Let $f:X\to R$ be a differentiable fun -ction defined on a nonempty subset X of R^n.

(a) If f is $B-(p,r,a)$ pseudo-invex function with respect to η and b on X, and k is any positive real number, then the function kf is $B-(p,\frac{r}{k},a)$ pseudo-invex functions with respect to the same η and b on X.

(b) If f is $B-(p,r,a)$ quasi-invex function with respect to η and b on X, and k is any positive real number, then the function kf is $B-(p,\frac{r}{k},a)$ quasi-invex functions with respect to the same functions η and b on X.

In following section, $B-(p,r,a)$-invex functions, $B-(p,r,a)$ quasi- invex functions and $B-(p,r,a)$ pesudo -invex functions are discussed only when $p\neq 0, r\neq 0$, other cases will be deal with likewise because the only changes arise from the form of inequality. The proofs in the other cases are easier than in this one. Moreover,with -out limiting generality of considerations, we shall assu -me that $r>0$ (in the case when $r<0$, the direction of some of the inequalities in the proofs of theorems should be changed to the opposite one).

III. Mond-weir Type Duality

In this section, we consider Mond–weir type dual and establish some duality results for multiobjective problem in which corresponding functions belong to classes of $B-(p,r,a)$-invex functions, $B-(p,r,a)$ quasi-invex func -tions, $B-(p,r,a)$ pseudo-invex functions with respect to η and b.

We consider below vector programming

$$(VP)\min\ f(x)=(f_1(x),\cdots,f_k(x))$$
$$s.t.\quad g(x)=(g_1(x),\cdots,g_m(x)), x\in X\subset R^n.$$

where $f_i(x):X\to R, i=1,\cdots,k$, $g_j(x):X\to R$, $j=1,\cdots,m$ are differentiable, its Mond-Weir dual progra -mming defined as below

$$(VD)\max\ f(y)=(f_1(y),\cdots,f_k(y))$$
$$s.t.\quad \sum_{i=1}^{k}\lambda_i\nabla f_i(y)+\sum_{j=1}^{m}\mu_j\nabla g_j(y)=0; \quad (1)$$
$$\sum_{j=1}^{m}\mu_j g_j(y)\ge 0; \quad (2)$$
$$\lambda=(\lambda_1,\cdots,\lambda_k)^T\ge 0,\ \mu=(\mu_1,\cdots,\mu_m)^T\ge 0.$$

Theorem 3.1 (Weak duality).Suppose that

(i) x is a feasible solution of (VP), (λ,μ,y) is a feas -ible solution of (VD);

(ii) $\sum_{i=1}^{k}\lambda_i f_i$ is $B-(p,r,a)$-invex function with respect to η and b_0 at y, $\sum_{j=1}^{m}\mu_j g_j$ is $B-(p,r,a)$ quasi- invex

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function with respect to η and b_1 at y ;

(iii) $b_0(x,y) > 0$ when $x \neq y,\ a(x,y)+c(x,y) \geq 0$.

Then $f(x) \leqq f(y)$ not hold.

Proof Since $g_j(x) \leq 0, \mu_j \geq 0,$ so $\sum_{j=1}^{m}\mu_j g_j(x) \leq 0,$ consider (2), we can get $\sum_{j=1}^{m}\mu_j g_j(x) \leq \sum_{j=1}^{m}\mu_j g_j(y),$ obviously have

$$\frac{1}{r}b_1(x,y)(e^{r(\sum_{j=1}^{m}\mu_j g_j(x)-\sum_{j=1}^{m}\mu_j g_j(y))}-1) \leq 0.$$

Using $\sum_{j=1}^{m}\mu_j g_j$ is $B-(p,r,a)$ -quasi-invex function with respect to η and b_1 at y , we have

$$\frac{1}{p}\sum_{j=1}^{m}\mu_j \nabla g_j(y)(e^{p\eta(x,u)}-I)+c(x,u) \leq 0, \qquad (3)$$

relation (1), (3) along with $a(x,y)+c(x,y) \geq 0$, we can get $\frac{1}{p}\sum_{i=1}^{k}\lambda_i \nabla f_i(y)(e^{p\eta(x,u)}-I)+a(x,u) \geq 0$, since $\sum_{i=1}^{k}\lambda_i f_i$ is $B-(p,r,a)$-invex function with respect to η and b_0 at y , we can get

$$\frac{1}{r}b_0(x,y)(e^{r(\sum_{i=1}^{k}\lambda_i(f_i(x)-f_i(y)))}-1) \geq 0.$$ by $b_0(x,y) > 0,$ we get $\sum_{i=1}^{k}\lambda_i(f_i(x)-f_i(y)) \geq 0,$ so $f(x) \leqq f(y)$ not hold.

Theorem 3.2 (Weak duality).Suppose that

(i) x is a feasible solution of (VP), (λ,μ,y) is a feasible solution of (VD);

(ii) $\sum_{i=1}^{k}\lambda_i f_i + \sum_{j=1}^{m}\mu_j g_j$ is $B-(p,r,a)$-invex function with respect to η and b_0 at y ;

(iii) $b_0(x,y) > 0$ when $x \neq y,\ a(x,y) > 0$.

Then $f(x) \leqq f(y)$ not hold.

Proof Suppose $f(x) \leqq f(y)$, then there exists $\lambda \in R^k{}_+$ such that $\sum_{i=1}^{k}\lambda_i f_i(x) \leq \sum_{i=1}^{k}\lambda_i f_i(y)$, also x is a feasible solution of (VP), (λ,μ,y) is a feasible solution of (VD), so there exists a $\mu \in R^m{}_+$ such that $\sum_{j=1}^{m}\mu_j g_j(x) \leq 0 \leq \sum_{j=1}^{m}\mu_j g_j(y)$ so, $\sum_{i=1}^{k}\lambda_i f_i(x) + \sum_{j=1}^{m}\mu_j g_j(x) \leq \sum_{i=1}^{k}\lambda_i f_i(y) + \sum_{j=1}^{m}\mu_j g_j(y)$. easily get

$$\frac{1}{r}b_0(x,y)(e^{r[(\sum_{i=1}^{k}\lambda_i(f_i(x)+\sum_{j=1}^{m}\mu_j g_j(x))-(\sum_{i=1}^{k}\lambda_i(f_i(y)+\sum_{j=1}^{m}\mu_j g_j(y))]}-1) \leq 0.$$

using $\sum_{i=1}^{k}\lambda_i f_i + \sum_{j=1}^{m}\mu_j g_j$ is $B-(p,r,a)$ -invex function with respect to η and b_0 at y , we have

$$\frac{1}{p}(\sum_{i=1}^{k}\lambda_i \nabla f_i(y) + \sum_{j=1}^{m}\mu_j \nabla g_j(y))(e^{p\eta(x,y)}-I) + a(x,y) \leq 0.$$

relation (1) along with $a(x,y) > 0$, we can get a contradiction, so $f(x) \leqq f(y)$ not hold.

Lemma 3.3[4] We say that *g* satisfies the generalized Slater type constraint qualification at a feasible point *x* if there exists a feasible point *x* such that *g*(*x*) < 0.

Lemma 3.4[4] Suppose that x is an efficient solution of (VP), assume that the Slater type constraint qualification is satisfied at x . Then, there exist $\lambda \in R^k, \lambda \geq 0$, $\mu \in R^m, \mu \geq 0,$ such that

$$\sum_{i=1}^{k}\lambda_i \nabla f_i(x) + \sum_{j=1}^{m}\mu_j \nabla g_j(x) = 0;$$

$$\sum_{j=1}^{m}\mu_j g_j(x) = 0.$$

Theorem 3.5(Strong duality).Suppose that x is an efficient solution of (VP), (λ^0,μ^0,y) is a feasible solution of (VD), and the generalized Slater type constraint qualification is satisfied at x , then exist $\lambda \in R^k, \lambda > 0,$ $\mu \in R^m,$ $\mu \geq 0,$ such that (λ,μ,x) is a feasible solution of (VD) and the objective functions of (VP) and (VD) are equal at x . If the hypotheses of the weak duality theorem 3.1 are fulfilled, then (λ,μ,x) is a efficient solution of (VD).

Proof from lemma 3 we can get $\exists \lambda \in R^k, \lambda \geq 0$, $\mu \in R^m, \mu \geq 0,$ such that(4), (5) hold, so (λ,μ,x) is a feasible solution of (VD), from result of theorem 3.1, we can get $f(x) \leqq f(y)$ not hold for all feasible solution of (VD), so (λ,μ,x) is an efficient solution of (VD).

Theorem 3.6(Strict Duality). Suppose that

(i) x and (λ,μ,y) be efficient solutions of problems (VP) and (VD), respectively;

(ii) $\sum_{i=1}^{k}\lambda_i f_i$ is $B-(p,r,a)$ invex with respect to η and

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b_0 at y, and $\sum_{j=1}^{m}\mu_j g_j$ is $B\text{–}(p,r,c)$ quasi-invex function with respect to η and b_1 at y;

(iii) $b_0(x,y)>0$ when $x\neq y$, $a(x,y)+c(x,y)\geq 0$.

Then $x=y$.

Proof Suppose that $x\neq y$, since x is an efficient solutions of (VP), so $\sum_{i=1}^{k}\lambda_i f_i(x)<\sum_{i=1}^{k}\lambda_i f_i(y)$ for λ (appear in (λ,μ,y)). easily get

$$\frac{1}{r}b_0(x,y)(e^{r(\sum_{i=1}^{k}\lambda_i(f_i(x)-f_i(y)))}-1)<0.$$ Also since $\sum_{i=1}^{k}\lambda_i f_i$ is $B\text{–}(p,r,a)$ -invex function with respect to η and b_0 at y, we can get

$$\frac{1}{p}\sum_{i=1}^{k}\lambda_i\nabla f_i(y)(e^{p\eta(x,u)}-I)+a(x,u)<0. \quad (4)$$

Since x and (λ,μ,y) be efficient solutions of problems (VP) and (VD), respectively, so $\sum_{j=1}^{m}\mu_j g_j(x)\leq 0\leq\sum_{j=1}^{m}\mu_j g_j(y)$ for μ (appear in (λ,μ,y)), so

$$\frac{1}{r}b_1(x,y)(e^{r(\sum_{j=1}^{m}\mu_j g_j(x)-\sum_{j=1}^{m}\mu_j g_j(y))}-1)\leq 0.$$

Also $\sum_{j=1}^{m}\mu_j g_j$ is $B\text{–}(p,r,a)$ quasi-invex function with respect to η and b_1 at y, we can get

$$\frac{1}{p}\sum_{j=1}^{m}\mu_j\nabla g_j(y)(e^{p\eta(x,u)}-I)+c(x,u)\leq 0,$$ relatio(1) along with $a(x,y)+c(x,y)\geq 0$, we can get

$$\frac{1}{p}\sum_{i=1}^{k}\lambda_i\nabla f_i(y)(e^{p\eta(x,u)}-I)+a(x,u)\geq 0,$$ it is a contradiction with (4), so $x=y$.

Theorem 3.7(Converse Duality). Suppose that

(i) x and (λ,μ,y) be efficient solutions of problems (VP) and (VD), respectively;

(ii) any one of the following conditions is satisfied:

(a) $\sum_{i=1}^{k}\lambda_i f_i$ is strictly $B\text{–}(p,r,a)$ invex with respect to η and b_0 at y, and $\sum_{j=1}^{m}\mu_j g_j$ is $B\text{–}(p,r,c)$ quasi-invex function with respect to η and b_1 at y;

(b) $\sum_{i=1}^{k}\lambda_i f_i+\sum_{j=1}^{m}\mu_j g_j$ is strictly $B\text{–}(p,r,a)$ pseudo -invex function with respect to η and b_0 at y;

(iii) $b_0(x,y)>0$ when $x\neq y$, $a(x,y)+c(x,y)\geq 0$.

Then y be efficient solutions in problems (VP).

Proof Assume that the condition (a) is fulfilled. We proceed by contradiction. Suppose that y is not an effici -ent solution of (VP). Then there exists a feasible soluti- on of (VP) x such that $\sum_{i=1}^{k}\lambda_i(x)\leq\sum_{i=1}^{k}\lambda_i f_i(y)$, for λ (appear in (λ,μ,y)), easily get

$$\frac{1}{r}b_0(x,y)(e^{r(\sum_{i=1}^{k}\lambda_i(f_i(x)-f_i(y)))}-1)\leq 0.$$

Also $\sum_{i=1}^{k}\lambda_i f_i$ is strictly $B\text{–}(p,r,a)$- invex with respect to η and b_0 at y, we can get

$$\frac{1}{p}\sum_{i=1}^{k}\lambda_i\nabla f_i(y)(e^{p\eta(x,u)}-I)+a(x,u)<0.$$

Since x and (λ,μ,y) be efficient solutions of probl- ems (VP) and (VD), respectively, so $\sum_{j=1}^{m}\mu_j g_j(x)\leq 0\leq\sum_{j=1}^{m}\mu_j g_j(y)$ for μ (appear in (λ,μ,y)), so

$$\frac{1}{r}b_1(x,y)(e^{r(\sum_{j=1}^{m}\mu_j g_j(x)-\sum_{j=1}^{m}\mu_j g_j(y))}-1)\leq 0.$$

Also $\sum_{j=1}^{m}\mu_j g_j$ is $B\text{–}(p,r,a)$-quasi-invex function with respect to η and b_1 at y, we can get

$$\frac{1}{p}\sum_{j=1}^{m}\mu_j\nabla g_j(y)(e^{p\eta(x,u)}-I)+c(x,u)\leq 0,$$ relation (1) along with $a(x,y)+c(x,y)\geq 0$, we can get

$$\frac{1}{p}\sum_{i=1}^{k}\lambda_i\nabla f_i(y)(e^{p\eta(x,u)}-I)+a(x,u)\geq 0,$$ it is a contradiction with (4), so y be efficient solutions of prob -lems (VP).

When the condition (b) is fulfilled. We proceed by con -tradiction. If y isn't an efficient solution of (VP), there exists a feasible solution of (VP) x, $x\neq y$ such that $\sum_{i=1}^{k}\lambda_i(x)\leq\sum_{i=1}^{k}\lambda_i f_i(y)$,by $\sum_{j=1}^{m}\mu_j g_j(x)\leq 0$, and $0\leq\sum_{j=1}^{m}\mu_j g_j(y)$, we have

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$$\sum_{i=1}^{k}\lambda_i f_i(x)+\sum_{j=1}^{m}\mu_j g_j(x)\le$$

$\sum_{i=1}^{k}\lambda_i f_i(y)+\sum_{j=1}^{m}\mu_j g_j(y)$,by $\sum_{i=1}^{k}\lambda_i f_i+\sum_{j=1}^{m}\mu_j g_j$ is strictly $B\text{–}(p,r,a)$-pesudo-invex function with respect to η and b_0 at y , we obtain

$$\frac{1}{p}(\sum_{i=1}^{k}\lambda_i\nabla f_i(y)+\sum_{j=1}^{m}\mu_j\nabla g_j(y))(e^{p\eta(x,y)}-I)$$
$$+a(x,y)<0$$

It's a contradiction with (2) and (iii). Thus, y is an effic –ient solution of (VP).

Theorem 3.8(strictly converse duality) Suppose that

(i) x is a feasible solution of (VP), (λ,μ,y) is a feasible solution of (VD);

(ii) $\lambda^{\mathrm{T}}f(x)\le\lambda^{\mathrm{T}}f(y)+\mu^{\mathrm{T}}g(y)$, $\sum_{i=1}^{k}\lambda_i f_i+\sum_{j=1}^{m}\mu_j g_j$ is strictly $B\text{–}(p,r,a)$ pesudo-invex function with respect to η and b_0 at y ;

(iii) $a(x,y)>0$, $b_0(x,y)>0$ when $x\ne y$.

Then $x=y$ and y is an efficient solution of (VD).

Proof Suppose that $x\ne y$,accordering to $\sum_{i=1}^{k}\lambda_i f_i+\sum_{j=1}^{m}\mu_j g_j$ is strictly $B\text{–}(p,r,a)$ pesudo-invex function with respect to η and b_0 at y , we can get

$$\frac{1}{r}b_0(x,y)(e^{r[(\sum_{i=1}^{k}\lambda_i(f_i(x)+\sum_{j=1}^{m}\mu_j g_j(x))-(\sum_{i=1}^{k}\lambda_i(f_i(y)+\sum_{j=1}^{m}\mu_j g_j(y))]}$$
$$-1)\ge\frac{1}{p}(\sum_{i=1}^{k}\lambda_i\nabla f_i(y)+\sum_{j=1}^{m}\mu_j\nabla g_j(y))$$
$$(e^{p\eta(x,y)}-I)+a(x,y)>0$$

by $b_0(x,u)>0$, we can get

$$\sum_{i=1}^{k}\lambda_i f_i(x)+\sum_{j=1}^{m}\mu_j g_j(x)-\sum_{i=1}^{k}\lambda_i f_i(y)-$$

$\sum_{j=1}^{m}\mu_j g_j(y)>0$,consider $\sum_{j=1}^{m}\mu_j g_j(x)\le 0$, so

$\sum_{i=1}^{k}\lambda_i f_i(x)>\sum_{i=1}^{k}\lambda_i f_i(y)+\sum_{j=1}^{m}\mu_j g_j(y)$, it's a cont -radiction with (ii), so $x=y$.

If x isn't an efficient solution of (VP), then follow the proof of [Theorem 3.7], we can obtain x is an effici -ent solution of (VP).

IV. CONCLUSION

In this paper, we introduce new classes of generalized invex function, that is, classes of $B\text{–}(p,r,a)$-invex func- tions, $B\text{–}(p,r,a)$ quasi-invex functions, $B\text{–}(p,r,a)$ pseudo -invex functions and strictly $B\text{–}(p,r,a)$ pesudo-invex functions, establish Mond-Weir dual problem multi- obje -ctive programming in which corresponding functions be -long to the introduced classes of functions, obtain many duality conditions under weaker convexity, which extend many results of [4].

Finally, duality problems of minimax fractional prog- ramming involving the introduced functions should be considered, Wolfe dual problem also should be consider -ed in the future.

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