The Self-Adaptive Multi-Splitting Parallel Methods for Non-Hermitian Positive Definite Systems

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***Abstract*— We present the convergent splitting and convergent multisplitting for linear system of algebraic equations $Ax = b$ when the coefficient matrix is a non-hermitian positive definite matrix. Furthermore, we also establish the comparison theorems of different splittings or multisplittings based on numerical radius. Mainly, we propose two new self-adaptive multisplitting parallel methods which the weighting matrices are self-adaptive. Finally, we give an application to solve the two- dimensional advection-diffusion equations.**

***Index Terms*— Non-hermitian matrix, positive definite matrix, multisplitting, comparison theorem**

AMS Subject Classifications: 65F10, 65F50

I. Introduction

Iterative methods based on matrix splittings play an important role for solving large sparse systems of linear equations (see [8] and [20]). The convergence property of a matrix splitting determines the numerical behaviour of the corresponding iteration method. Therefore, to study various typical matrix splitting is much important in the modern matrix iterative analysis. Weak regular splitting and regular splitting for a monotone matrix (see [2], [8], [20]), and P-regular splitting for a hermitian positive definite matrix (see [8,14,20]), are well known concepts of matrix splittings, and they have widely employed in the convergence analysis of various matrix splitting methods, e.g., the classical relaxation methods [8, 20], the parallel matrix multisplitting methods [1,4-7,9,10,15-17], and the asynchronous parallel matrix multisplitting methods [2,3,19], for solving the systems of linear equations. However, a little is known whether a splitting of a non-hermitian positive definite matrix is convergent or not. Therefore, few matrix splitting methods are used for solving the systems of linear equations when the coefficient matrix is a non-hermitian positive definite matrix, many authors focus on *polynomial iterative methods* [13]. Wang and Bai [18] gave some sufficient conditions which guarantee that a single splitting is a convergent splitting. In this paper, we present some convergent splittings and multisplittings under suitable conditions and establish some comparison theorems. As we know, the weighting matrices are given in advance, they are not known to be good or bad, this influences the efficiency of parallel methods. In fact, none has ever studied that how to choose optimal weighting matrices, we also discuss this problem in the paper.

Here, we still use the scalar weighting matrices $E_i^{(k)} = \alpha_i^{(k)} I$, $(i = 1, 2, \cdots, m, k = 1, 2, \cdots)$ for the nonsingular matrix A , but $\alpha_i^{(k)}, (i = 1, 2, \cdots, m, \ \ k = 1, 2, \cdots)$ are chosen by finding the best point in the hyperplane H_k, where

$$H_k = \{x | x = \sum_{i=1}^{m} \alpha_i^{(k)} x_i^{(k)}, \ \ \sum_{i=1}^{m} \alpha_i^{(k)} = 1\}$$

$$k = 1, 2, \cdots \quad (1)$$

Thus, $\alpha_i^{(k)} (i = 1, 2, \cdots, m, \ k = 1, 2, \cdots)$ are the optimal parameters in k-th iteration. In other words, the point $x^{(k)} = \sum_{i=1}^{m} \alpha_i^{(k)} x_i^{(k)}$ generated by the optimal weighting matrices $E_i^{(k)} = \alpha_i^{(k)} I, (i = 1, 2, \cdots, m)$, $(k = 1, 2, \cdots)$ may be the projection of the solution of linear systems in H_k. Obviously, the optimal weighting matrices are different from the original which are nonnegative (see [14,16]).

We firs give some essential notations and preliminaries. We use $C^{n\times n}$ to denote the $n \times n$ complex matrix set, and C^n the n-dimensional complex vector set. In particular, we use $R^{n\times n}$ to denote the $n \times n$ real matrix set, and R^n the n-dimensional real vector set. For an $x \in C^n$, x^* is used to represent the conjugate transpose of the vector x. For a matrix $A \in C^{n\times n}$, $H(A)$ and $S(A)$ are used to denote the hermitian and skew-hermitian parts of matrix A, respectively, i.e, $H(A) = \frac{A^*+A}{2}$ and $S(A) = \frac{A-A^*}{2}$. Moreover, we use $r(A), \rho(A)$, and $\lambda(A)$ to denote the numerical radius (see [12]), the spectral radius and the spectrum of the matrix A, respectively.

We call $A \in C^{n\times n}$ a positive definite matrix if its hermitian part $H(A)$ is hermitian positive definite (denoted by $H(A) \succ 0$). Evidently, $A \in C^{n\times n}$is a hermitian positive definite matrix if and only if $A = H(A)$. $A = M - N$ is called a splitting of the matrix $A \in C^{n\times n}$ if $M \in C^{n\times n}$ is nonsingular. This

This is supported by National Science Found of China (11071184) and National Science Found of Shanxi Province (2010011006)

doi:10.4304/jsw.7.11.2400-2404

splitting is called a convergent splitting if $\rho(M^{-1}N) < 1$.

II. The Convergent Splitting

To consider the large sparse linear system of algebraic equations

$$Ax = b, \tag{2}$$

where A is a non-hermitian positive definite matrix.

Theorem 2.1 Let A be a positive definite matrix, and let $A = M - N$ be a splitting. Then $\rho(M^{-1}N) < 1$ if $M + N$ is a hermitian positive definite matrix.

Proof. Let $B = M + N = M - (-N)$. Then the splitting $(M, -N)$ of the matrix B is a P-regular splitting. Hence, we obtain that $\rho(M^{-1}N) < 1$. □.

Theorem 2.2 Let A be a positive definite matrix, and let $A = M - N$ be a splitting. Then $\rho(M^{-1}N) < 1$ if N is a hermitian matrix and $M + N$ is positive definite.

Proof. Let λ be the maximal eigenvalue of the iteration matrix $M^{-1}N$ and v be the corresponding eigenvector. Then

$$M^{-1}Nv = \rho(M^{-1}N)v.$$

It is evident that

$$\begin{aligned}&\rho(M^{-1}N) = |\frac{v^*Nv}{v^*Mv}|\\ &\le \max_{x\in C^n} |\frac{x^*Nx}{x^*Mx}|\\ &= \max_{x\in C^n} |\frac{x^*Nx}{x^*H(A)x + x^*S(A)x + x^*Nx}|\\ &\le \max_{x\in C^n} |\frac{x^*Nx}{x^*H(A)x + x^*Nx}|\\ &\le 1.\end{aligned}$$

Which completes the proof of this theorem. □

But two kinds of splittings in Theorem 2.1 and Theorem 2.2 can not be generalized to multisplitting of A. Hence, we give some conditions of $S(A)$.

Let U be a unitary matrix and satisfy

$$S(A) = U^*\Lambda U,$$

where Λ is a diagonal matrix.

Let $|S(A)| = U^*|\Lambda|U$. Obviously, $|S(A)|$ is a hermitian positive semi-definite matrix. Here, we suppose that the following condition holds in later sections without special explain.

$$H(A) - |S(A)| \succ 0 \tag{3}$$

Theorem 2.3 Let $A = M - N$ be a splitting. Assume that M is hermitian and $M + M^* \succ H(A) + |S(A)|$. Then the splitting is a convergent splitting.

Proof. It is known that we only need to show that $\rho(M^{-1}N) < 1$.

From Theorem 2.2, we obtain that

$$\begin{aligned}&\rho(M^{-1}N) \le \max_{x\in C^n} |\frac{x^*Nx}{x^*Mx}|\\ &\le \max_{x\in C^n} |\frac{x^*S(A)x + x^*(M - H(A))x}{x^*H(A)x + x^*(M - H(A))x}|\\ &= \max_{y^*y=1} |\frac{y^*H^{-\frac{1}{2}}S(A)H^{-\frac{1}{2}}y + y^*H^{-\frac{1}{2}}MH^{-\frac{1}{2}}y - 1}{y^*H^{-\frac{1}{2}}MH^{-\frac{1}{2}}y}|.\end{aligned}$$

Let

$$f(t) = \frac{\sqrt{s^2 + (t-1)^2}}{t}.$$

Then $f(t) < 1$ if and only if

$$2t > 1 + s^2. \tag{4}$$

We replace s, t by $y^*H^{-\frac{1}{2}}S(A)H^{-\frac{1}{2}}y,\ y^*H^{-\frac{1}{2}}MH^{-\frac{1}{2}}y$ respectively. (4) implies

$$2y^*H^{-\frac{1}{2}}MH^{-\frac{1}{2}}y > 1 + |y^*H^{-\frac{1}{2}}S(A)H^{-\frac{1}{2}}y|^2. \tag{5}$$

Based on (3), (5) holds if the following inequality holds

$$2y^*H^{-\frac{1}{2}}MH^{-\frac{1}{2}}y > 1 + \{y^*H^{-\frac{1}{2}}|S(A)|H^{-\frac{1}{2}}y\}^2. \tag{6}$$

From $y^*y = 1$, (6) holds if

$$2M \succ H + |S(A)|H^{-1}|S(A)|.$$

From (3), we have $H^{-1} \prec |S(A)|^{-1}$. Hence, $2M \succ H + |S(A)|H^{-1}|S(A)|$ holds if $M + M^* \succ H(A) + |S(A)|$ holds. Thus, We obtain

$$\rho(M^{-1}N) < 1,$$

which completes the proof of this theorem. □

Obviously, if $|S(A)| = 0$, the condition $M + M^* \succ H(A)$ of Theorem 2.3 is that of a P-regular spitting. But if $|S(A)| \ne 0$, this condition is sufficient.

III. The Comparison Theorems

In this section, we give the comparison theorems of different splittings.

Let P be a semi-hermitian positive definite matrix and satisfy

$$x^*Px \cdot x^*H(A)x \ge -(x^*S(A)x)^2. \tag{7}$$

Theorem 3.1 Let $A = M_1 - N_1 = M_2 - N_2$ be two different splittings, and $M_1 \succeq M_2 \succeq H(A) + P$, then $\rho(M_1^{-1}N_1) \ge \rho(M_2^{-1}N_2)$.

Proof. To obtain the following inequality

$$\rho(M_1^{-1}N_1) \ge \rho(M_2^{-1}N_2),$$

we only need to prove $\forall x \in C^n$,

$$|\frac{x^*N_1x}{x^*M_1x}| \ge |\frac{x^*N_2x}{x^*M_2x}|.$$

Which is written as follows,

$$\begin{aligned}&|\frac{x^*S(A)x + x^*(M_1 - H(A))x}{x^*H(A)x + x^*(M_1 - H(A))x}|\\ &\ge |\frac{x^*S(A)x + x^*(M_2 - H(A))x}{x^*H(A)x + x^*(M_2 - H(A))x}|.\end{aligned} \tag{8}$$

In order to obtain above inequality, we observe that the function

$$f(y) = \frac{\sqrt{s^2 + y^2}}{h + y}.$$

Differentiating $f(y)$ leads to

$$f'(y) = \frac{yh - s^2}{(h + y)^2\sqrt{s^2 + y^2}}. \tag{9}$$

$f'(y) \geq 0$ if $yh \geq s^2$.
We replace h, s, y by $x^*H(A)x, |x^*S(A)x|$, $x^*(M - H(A))x$. The condition $M_1 \succeq M_2 \succeq H(A) + P$ guarantees (7). Obviously, the inequality (7) guarantees the inequality (8). Which completes the proof of this theorem. □

Theorem 3.2 Let $A = M_1 - N_1 = M_2 - N_2$ be two different splittings. Assume that $\frac{1}{2}(H(A)+|S(A)|) \prec M_1 \preceq M_2 \preceq H(A)$ holds. Then $\rho(M_1^{-1}N_1) \geq \rho(M_2^{-1}N_2)$.

Proof. The condition $\frac{1}{2}(H(A) + |S(A)|) \prec M_1 \preceq M_2$ guarantees the two splittings are convergent splittings. We prove that

$$\rho(M_1^{-1}N_1) \geq \rho(M_2^{-1}N_2).$$

We know from (9) that $f'(y) \leq 0$ if $yh \leq s^2$. Thus, the same analysis as Theorem 3.1, we obtain the conclusion. □

IV. Self-Adaptive Multispliting Parallel Methods

In this section, we present three multisplitting parallel algorithms, one is an application of usually multisplitting parallel algorithm to the non-hermitian linear systems (see [16]), others are the new multisplititng parallel algorithm with self-adaptive weighting matrices, and analyze the convergence of these multisplitting parallel algorithms for solving the linear systems (2).

Let $A = M_i - N_i, \quad i = 1, 2, \cdots, m$.

Algorithm 4.1 Give an initial point $x^{(0)}$, a precision $\epsilon > 0$ and the reasonable parameters $\alpha_i,\ i = 1, 2, \cdots, m$. For $k = 0, 1, 2, \cdots,$ until converges.

Step 1

$$M_i x_i^{(k)} = N_i x^{(k-1)} + b. \tag{10}$$

Step 2

$$x^{(k)} = \sum_{i=1}^{m} E_i x_i^{(k)}, \tag{11}$$

where $E_i = \alpha_i I \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$.

Step 3 If $||Ax^{(k)} - b|| \leq \epsilon$, stop; Otherwise, goto step 1.

Algorithm 4.2 Give an initial point $x^{(0)}$, a precision $\epsilon > 0$. For $k = 0, 1, 2, \cdots,$ until converges.

Step 1

$$M_i x_i^{(k)} = N_i x^{(k-1)} + b. \tag{12}$$

Step 2 Let $r_i^{(k)} = Ax_i^{(k)} - b, \quad r = \sum_{l=1}^{m(k)} \alpha_i r_i^{(k)}$.

$$\min_{\alpha} r^T r$$
$$s.t. \qquad \sum_{i=1}^{m(k)} \alpha_i = 1.$$

Step 3

$$x^{(k)} = \sum_{i=1}^{m} \alpha_i x_i^{(k)}. \tag{13}$$

Step 4 If $||Ax^{(k)} - b|| \leq \epsilon$, stop; Otherwise, goto step 1.

Introducing the matrices

$$G = \sum_{k=1}^{m} E_k M_k^{-1}, \qquad T = \sum_{k=1}^{m} E_k M_k^{-1} N_k, \tag{14}$$

we can express, if G is nonsingular, the multisplitting $(M_k, N_k, E_k)_{k=1}^m$ as a single splitting $(G^{-1}, G^{-1}T)$ of A.

Theorem 4.1 Let $A = M_i - N_i\ (i = 1, 2, \cdots, m)$ be a multisplitting of A. If M_i are hermitian and $M_i + M_i^* \succ H(A) + |S(A)|$. Then $G^{-1} + G^{-*} \succ H(A) + |S(A)|$.

Proof. Since each M_k is positive definite, from Theorem 2.5 of [3], G is also positive definite and in particular nonsingular. To obtain $G^{-1} + G^{-*} \succ H(A) + |S(A)|$ if and only if

$$\begin{aligned} &G^*(G^{-1} + G^{-*} - (H(A) + |S(A)|))G \\ &= G^* + G - G^*(H(A) + |S(A)|)G \succ 0. \end{aligned}$$

Let $Q_k = M + M^* - H(A) - |S(A)|$, then $Q_k \succ 0$.
Now, we use essentially the same proof as Nabben [14, Theorem 3.2]:

$$\begin{aligned}
&G + G^* - G^*(H(A) + |S(A)|)G \\
&= \sum_{k=1}^{m} (E_k M_k^{-1} + M_k^{-*} E_k) \\
&- \sum_{k,j=1}^{m} M_k^{-*} E_k (H(A) + |S(A)|) E_j M_j^{-1} \\
&= \sum_{k=1}^{m} M_k^{-*} (M_k^* E_k + E_k M_k) M_k^{-1} \\
&- \sum_{k=1}^{m} M_k^{-*} (E_k (H(A) + |S(A)|) E_k) M_k^{-1} \\
&- \sum_{k,j=1,k\neq j}^{m} M_k^{-*} E_k (H(A) \\
&+ |S(A)|) E_j M_j^{-1} \\
&= \sum_{k=1}^{m} M_k^{-*} (E_k Q_k \\
&+ \sum_{j=1,k\neq j}^{m} E_k (H(A) + |S(A)|) E_j M_k^{-1} \\
&- \sum_{k,j=1,k\neq j}^{m} M_k^{-*} E_k (H(A) + |S(A)|) E_j M_j^{-1} \\
&= \sum_{k=1}^{m} \alpha_k M_k^{-*} Q_k M_k^{-1} \\
&+ \sum_{k,j=1,k\neq j}^{m} \alpha_k \alpha_j M_k^{-*} (H(A) + |S(A)|) \\
&(M_k^{-1} - M_j^{-1}) \\
&= \sum_{k=1}^{m} \alpha_k M_k^{-*} Q_k M_k^{-1} \\
&+ \tfrac{1}{2} \sum_{k,j=1,k\neq j}^{m} \alpha_k \alpha_j (M_k^{-*} - M_j^{-*}) \\
&(H(A) + |S(A)|)(M_k^{-1} - M_j^{-1})
\end{aligned}$$

Thus, the first sum is hermitian positive definite, the second sum is positive semi-definite. Hence, we have completed the proof of this theorem. □

Theorem 4.2 Let $A = M_i - N_i\ (i = 1, 2, \cdots, m)$ be a multisplitting of A. If M_i are hermitian and $M_i + M_i^* \succ H(A) + |S(A)|$. Then the splitting $A = G^{-1} - G^{-1}T$ is a convergent splitting.

Proof. We know from Theorem 2.1 and Theorem 4.1 that the splitting $A = G^{-1} - G^{-1}T$ is a convergent splitting. □

To compare the asymptotic convergence rate of multisplitting parallel methods resulted from two different multisplittings $(M_k, N_k, E_k)_{k=1}^m$ and $(\bar{M}_k, \bar{N}_k, E_k)_{k=1}^m$ of A, we construct matrices

$$\bar{G} = \sum_{k=1}^{m} E_k \bar{M}_k^{-1}, \quad \bar{T} = \sum_{k=1}^{m} E_k \bar{M}_k^{-1} \bar{N}_k, \tag{15}$$

Theorem 4.3 Let $A = M_i - N_i = \bar{M}_i - \bar{N}_i (i = 1, 2, \cdot, m)$ be splittings of A. Assume that $M_i \succeq \bar{M}_i \succeq H(A) + P, i = 1, 2, \cdots, m$ hold. Then $\rho(T) \geq \rho(\bar{T})$.

Proof. From Theorem 3.1 we know that we only need to show

$$G^{-1} \succeq \bar{G}^{-1} \succeq H(A) + P. \tag{16}$$

We obtain from Theorem 4.1 and Theorem 3.1 that (16) holds. □

Theorem 4.4 Let $A = M_i - N_i = \bar{M}_i - \bar{N}_i$ $(i = 1, 2, \cdot, m)$ be splittings of A. Assume

$$\frac{1}{2}(H(A) + |S(A)|) \prec M_i \preceq \bar{M}_i \preceq H(A), \ i = 1, \cdots, m.$$

Then $\rho(T) \geq \rho(\bar{T})$.

Proof. We can obtain from Theorem 4.1 and Theorem 3.2 that $\rho(T) \geq \rho(\bar{T})$. □

Theorem 4.5 Let $A = M_1 - N_1 = H - (-S)$, and let $A = M_i - N_i$ $(i = 2, \cdots, m)$ be splittings of A. If $||SH^{-1}||_2 < 1$, then $x^{(k)}$ generated by Algorithm 4.2 converges to the solution of linear systems (2).

Proof. Let x^* be the unique solution for system linear equations (2). If we write

$$\varepsilon_i^{(k)} = x_i^{(k)} - x^*, i = 1, 2, \cdots, m,$$

then

$$\varepsilon^{(k)} = x^{(k)} - x^*.$$

From Algorithm 4.2, we have

$$\begin{aligned} ||r^{(k)}||_2 &\leq ||r_1^{(k)}||_2 \\ &= ||A\varepsilon_1^{(k)}||_2 \\ &= ||AM_1^{-1}N_1\varepsilon^{(k-1)}||_2 \\ &= ||AM_1^{-1}N_1A^{-1}A\varepsilon^{(k-1)}||_2 \\ &\leq ||AM_1^{-1}N_1A^{-1}||_2 \times ||A\varepsilon^{(k-1)}||_2 \\ &= ||N_1M_1^{-1}||_2 \times ||r^{(k-1)}||_2 \\ &= ||SH^{-1}||_2 \times ||r^{(k-1)}||_2 \\ &\cdots \quad \cdots \\ &= \Pi\beta^k ||A\varepsilon^{(0)}||_2 \\ &= \Pi\beta^k ||r^{(0)}||_2, \end{aligned}$$

where $\beta = ||SH^{-1}||_2$.

From the assumption $\beta < 1$, we have

$$\lim_{k\to\infty} r^{(k)} = 0.$$

Thus, we have completed the proof of the theorem.

Algorithm 4.3 Give an initial point $x^{(0)}$, a precision $\epsilon > 0$. For $k = 0, 1, 2, \cdots$, until converges.

Step 1

$$M_i x_i^{(k)} = N_i x^{(k-1)} + b, \tag{17}$$

Step 2 Let $r_i^{(k)} = Ax_i^{(k)} - b, \quad r = \sum_{l=1}^{m(k)} \alpha_i r_i^{(k)}$.

$$\begin{aligned} &\min_{\alpha} r^T H^{-1} r \\ s.t. \quad &\sum_{i=1}^{m(k)} \alpha_i = 1. \end{aligned}$$

Step 3

$$x^{(k)} = \sum_{i=1}^{m} \alpha_i x_i^{(k)}. \tag{18}$$

Step 4 If $||Ax^{(k)} - b|| \leq \epsilon,$ stop; Otherwise, goto step 1.

Theorem 4.6 Let $A = M_1 - N_1 = H - (-S)$, and let $A = M_i - N_i (i = 2, \cdots, m)$ be splittings of A. If (3) holds, then $x^{(k)}$ generated by Algorithm 4.3 converges to the solution of linear systems (2).

Proof. Let x^* be the unique solution for system linear equations (2). If we write

$$\varepsilon_i^{(k)} = x_i^{(k)} - x^*, i = 1, 2, \cdots, m,$$

then

$$\varepsilon^{(k)} = x^{(k)} - x^*.$$

From the Algorithm 4.3, we have

$$\begin{aligned} || \quad & H^{-\frac{1}{2}} r^{(k)}||_2 \leq ||H^{-\frac{1}{2}} r_1^{(k)}||_2 \\ = \quad & ||H^{-\frac{1}{2}} A\varepsilon_1^{(k)}||_2 \\ = \quad & ||H^{-\frac{1}{2}} AM_1^{-1}N_1\varepsilon^{(k-1)}||_2 \\ = \quad & ||H^{-\frac{1}{2}} AM_1^{-1}N_1A^{-1}H^{\frac{1}{2}}H^{-\frac{1}{2}}A\varepsilon^{(k-1)}||_2 \\ \leq \quad & ||H^{-\frac{1}{2}} AM_1^{-1}N_1A^{-1}H^{\frac{1}{2}}||_2 \times ||H^{-\frac{1}{2}}A\varepsilon^{(k-1)}||_2 \\ = \quad & ||H^{-\frac{1}{2}} N_1M_1^{-1}||_2 \times ||H^{-\frac{1}{2}} r^{(k-1)}||_2 \\ = \quad & ||H^{-\frac{1}{2}} SH^{-\frac{1}{2}}||_2 \times ||H^{-\frac{1}{2}} r^{(k-1)}||_2 \\ \cdots \quad & \cdots \\ = \quad & \Pi\beta^k ||H^{-\frac{1}{2}} A\varepsilon^{(0)}||_2 \\ = \quad & \Pi\beta^k ||H^{-\frac{1}{2}} r^{(0)}||_2, \end{aligned}$$

where $\beta = ||H^{-\frac{1}{2}} SH^{-\frac{1}{2}}||_2$.

From the assumption $\beta < 1$, we have

$$\lim_{k\to\infty} r^{(k)} = 0.$$

Thus, we have completed the proof of the theorem.

Consider the system of linear equations

$$(A + iB)(x_1 + Ix_2) = b_1 + ib_2, \tag{19}$$

where i denote the imaginary unit, A and B are hermitian positive definite matrices. The system of complex linear equations (19) can be equivalently written as

$$Wx = b, \quad W = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{20}$$

Thus, we obtain

$$H(W) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad S(W) = \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix},$$

when $A - B$ is a hermitian positive definite matrix, (3) holds.
Obviously,

$$|S(W)| \preceq \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

Thus, let $A + B = D - L_i - U_i (i = 1, 2, \cdots, m)$, with L being the strictly lower triangular matrices, $D = diag(A + B)$ and U not being the strictly upper triangular matrices. It is obtained that $M_i + M_i^* \succeq A + B$ if $M_i = D - L_i - L_i^T$. Hence, the multisplitting parallel method based on above multisplitting is convergent.

V. Numerical Example

we present numerical results to demonstrate our theoretical results for the two- dimensional advection-diffusion equations,

$$\begin{cases} -\nu \triangle \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{u} + \sigma \mathbf{u} = \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{u} = 0 & \text{on} \quad \Omega. \end{cases}$$

In all experiments here, we consider the advection-diffusion equations on $\Omega = [0, 1] \times [0, 1]$ with a body force $\mathbf{f}(x)$ such that the true solution is $\mathbf{u} = (u, v)$,

$$\begin{cases} u &= sin(\pi x) * sin(\pi y), \\ v &= (x^2 - x)(y^2 - y), \end{cases}$$

the convection field $\mathbf{b} = (1, 1)$ and with the parameters $\nu = 1, \ \sigma = 1$. Let T_h be a convention decomposition of Ω into uniform rectangular K. All the numerical experiments have been performed using the conforming Q_1 finite element $\mathbf{V}_h$,

$$\mathbf{V}_h = \{\mathbf{v}_h \in H^1(\Omega)^2 | \ \mathbf{v}_h|_K \in Q_1(K)^2, \forall K \in T_h\}.$$

We consider the standard finite element method, then the variational formulation of the above two-dimensional advection-diffusion equations is: find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\nu(\mathbf{u}_h, \mathbf{v}_h) + (\mathbf{b} \cdot \nabla \mathbf{u}_h, \mathbf{v}_h) + \sigma(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

The stepsizes along both x and y directions are the same, i.e., $h = \frac{1}{32}$.

Let the coefficient matrix A split into

$$A = D - L - U,$$

where $D = diag(A)$, L is the strictly lower triangle matrix and U is strictly upper triangle matrix.

We use the following three splittings:
(a) The Gauss-Seidel splitting

$$M_1 = D - L, \quad N_1 = U;$$

(b) The reversed Gauss-Seidel splitting

$$M_2 = D - U, \quad N_2 = L;$$

(c)The hermitian and skew- hermitian splitting

$$M_3 = H(A), \quad N_3 = -S(A).$$

The computation result of Algorithm 4.3 is in the following when the initial vector $x^{(0)} = 0$ and the precision $\epsilon = 10^{-5}$.

Table 5.1 Computational results of Algorithm 4.3

n		Algorithm 4.3
32	IT	6
	CPU(s)	0.3432

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