Applications of Matlab in Mathematical Analysis

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***Abstract*—In international academia, Matlab has already been accepted as an accurate and reliable standard computational soft ware. It is widely applied in applied algebra, statistics processing, automation and digital communication. Matlab has also been incorporated into the study of calculus. In this paper, we demonstrate Weierstrass function which has the property of being continuous everywhere but differentiable at no where, the graph of Weierstrass function is given by the use of Matlab. Besides, we study the numerical approximation of π by using Matlab and Romberg method.**

***Index Terms*—Matlab, Weierstrass function, Romberg method.**

I. INTRODUCTION

Currently, over thirty mathematics soft wares are widely applied. These mathematics soft wares can be divided into two categories in terms of their function. The first category including Mathematica and Maple features on mathematical analysis, symbolic computation, formula reasoning and are capable of computing the analytical solution to mathematical problems while the disadvantage of these software are that they are not efficient when it comes to processing large quantity of data. The second category is the numerical computing soft wares including Matlab, XPASS and GAUSS etc. The advantages of these softwares are that they are efficient to process large quantity of data. Nowadays, the core part of the first category software's have been incorporated into the second category, say Matlab, and that makes Matlab become the main stream software in the field of mathematics. In the international academic community, Matlab has been recognized as accurate and reliable scientific computing standard software. In many international academic journals (especially electronic information science journals), you can see the application of Matlab. In the design of research units and industry, Matlab is considered the first choice for research and development software tools.

In recent years, with the introduction of new media products, there has been a shift in the use of programming languages from FORTRAN or C to Matlab for implementing numerical methods. Over the years, many papers have been written on the subject of numerical methods. The main benefit is that one doesn't have to know the mathematical theory in order to apply the numerical methods for solving their real-life problems.

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Matlab stands for MATRIX LABORATARY (Matrix Laboratory), which was first used by Dr. CleveMoler in New Mexico University United States to teach courses of linear algebra. The basic data unit is a matrix without dimension restriction. In Matlab, the matrix computing becomes extremely easy. Dr. Moler in 1984 launched the official version of the software, in the later editions, he also gradually added to the control system, system identification, signal processing and communications, more than a toolbox, so Matlab became to be widely used in automatic control, image signal processing , biomedical engineering, speech processing, radar engineering, signal analysis, optimization and other fields. It has the following functions and features: efficient numerical computation and symbolic computation capabilities, enables us to shift from the analysis of complex mathematical operations freed to complete graphics, and from programming to achieve the results visualization; feature-rich application of the toolbox, providing a large number of convenient and practical processing tools; friendly interface and close to the mathematical expression of the naturalization language, easy to learn and master. Practice has proved that one can learn to tens of minutes to the basics of Matlab, after the initial few hours of use will be able to master it. Second, Matlab in communication theory teaches examples and analysis of the characteristics of the signal in the frequency domain. Communication theory is the study of signal transmission in communication systems, in many cases to analyze the characteristics of the signal. Method for determining the signal obtained by Fourier transform spectrum analysis of random signals is through its power spectral density. Analytical method used to analyze the signal, the signal spectrum can only be a function of expression, the expression under the function of artificial drawing is difficult to draw the diagram is also very accurate. But by the use of Matlab, it is easy to get the signal of the spectrum.

In pure mathematics, since Matlab is an integrated computer software which has three functions: symbolic computing、numerical computing and graphics drawing. Matlab is capable to carry out many functions including computing polynomials and rational polynomials, solving equations and computing many kind of mathematical expressions. One can also use Matlab to calculate the limit, derivative, integral and Taylor series of some mathematical expressions. With Matlab, The graphs of functions with one or two variables can be easily drawn in selected domain. Therefore, functions can be studied by visualization for their main Characteristics. Matlab is

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also a system which can be easily expanded. Matlab provides many powerful software packages which can be easily incorporated into the clients system.

Recently, there are many papers in the literature which are devoted to the application of Matlab in mathematical analysis, see [1][2][3].

In the section 2 of this paper, we will show the graph of the very interesting Weierstarass function by use of Matlab, to our best knowledge, the graph of this function has never been shown in literature. In section 3, we will try to approximate π numerically by the use of Matlab and Romboger method.

II. Graph of Weierstarass Function

In calculus, we usually consider the continuity and differentiability of functions. Mathematicians once tend to think that a continuous function should be differentiable nearly every where except at some discrete points. But with the development of series theory, Weierstrass put forward a counter example which shows that a continuous function can be indifferentiable every where in its domain(see [5][6][7]. The counter example is well constructed, but we never saw its graph in literature. Now, we try to draw its graph with Matlab. To understand this problem well, first, let us construct function

$$f(x)=\sum_{n=0}^{\infty}\frac{\varphi(10^n x)}{10^n},$$

Where $\varphi(x)$ stands for the distance between x and the closest integer. For example, if $x=1.27$, $\varphi(x)=0.27$; if $x=4.83$, $\varphi(x)=0.17$. Obviously, $\varphi(x)$ is a continuous function with periodic 1.

Based on the continuity of $\varphi(x)$, we can easily prove that $f(x)\in C(-\infty,+\infty)$.

We now consider the differentiability of $f(x)$. Let x be an arbitrary point in the domain. Since $f(x)$ is a periodic function, we may assume $0\le x<1$ without loss of generality. We write x in the form

$x=0.a_1a_2\cdots a_n\cdots$, Let

$$h_m=\begin{cases}10^{-m}, a_m=0,1,2,3,5,6,7,8,\\ -10^{-m}, a_m=4,9,\end{cases}.$$

Obviously, we have

$h_m\to 0(m\to\infty)$.

And we also have

$$\begin{cases}\varphi(10^n(x+h_m))=\varphi(10^n x\pm 10^{n-m})=\varphi(10^n x), n\ge m,\\ \varphi(10^n(x+h_m))-\varphi(10^n x)=\pm 10^n h_m, n<m.\end{cases}$$

Based on that, we have

$$\frac{f(x+h_m)-f(x)}{h_m}=\sum_{n=0}^{m-1}\pm 1.$$

The right side of the above is sure an integer and it's positive when m is odd negative when m is even. So we come to the conclusion

$\lim_{m\to\infty}\frac{f(x+h_m)-f(x)}{h_m}$ does not exist therefore $f(x)$ is not differentiable at x.

This function was provided by Dutch mathematician Van Der Waerden in 1930. But it is no easy to draw its graph. With the help of Matlab, we can try to do it.

Construct $\varphi(x)$ as follow;

```
function fff=z(x)
if x>round(x)
   fff=x-round(x)
elseif x<round(x)
   fff=round(x)-x
else x==round(x)
   fff=0
end
```

Construct $f(x)$ as follow;

```
function gg=f(x)
gg=z(x)
for i=1:1000
   gg=gg+z(10^i*x)/10^i;
end


function f=ws(n)
ws=fai(n)
for i=1:10000
   ws=ws+fai(10^i*n)/10^i;
end
```

Then we can use the plot function provided by Matlab to draw the graph of Weierstarass function. See fig1-fig4.

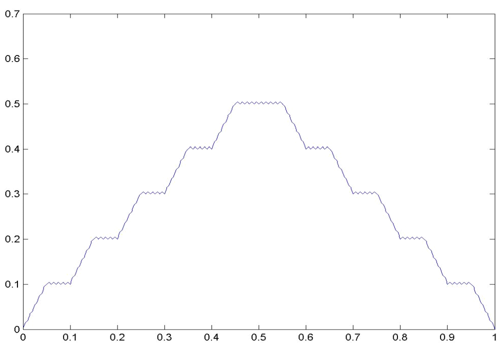

Fig1

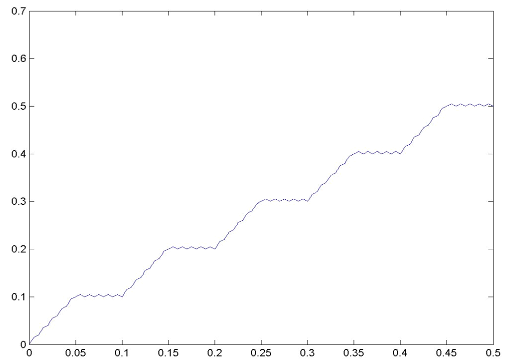

Fig 2

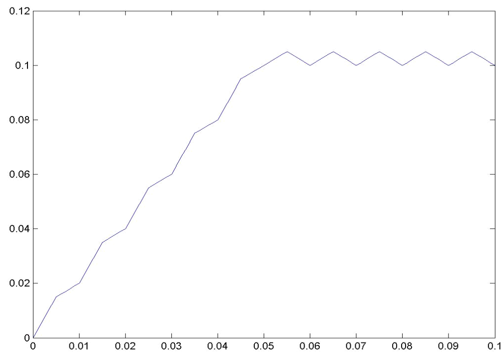

Fig 3

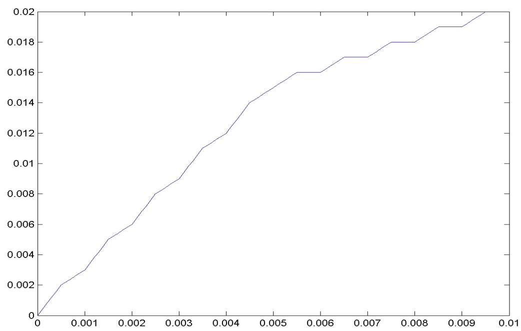

Fig4

Fig 1—4 shows the graph of Weierstrass function defined on [0,1], [0,0.5], [0,0.1]and [0,0.01]respectively. From the graph, we can see the function is a continuous but differentiable at nowhere.

III. Approximation of Circumference Ratio PI with Romberg Method

Circle is a shape with all points the same distance from the center. If you measure the distance around a circle and divide it by the distance across the circle through the center, you will always come close to a particular value, depending upon the accuracy of your measurement. This value is approximately 3.1415926.... We use the Greek letter π (pronounced Pi) to represent this value. The number π goes on forever. To calculate π using computers is always an interesting thing not only for calculation of π its self but also for testing software and the efficiency of a particular algorithm as well. In this section, we will try to approximate π by using Matlab and Romberg method (see [8][9][10][11][12]).

First of all, we recall Romberg method. In numerical analysis, Romberg's Method (Romberg 1955) generates a triangular array consisting of numerical estimates of the definite integral by applying Richardson extrapolation repeatedly on the Trapezium Rule or the Rectangle Rule. Romberg's method is a Newton–Cotes formula. It evaluates the integrand at equally-spaced points. The integrand must have continuous derivatives, though fairly good results may be obtained if only a few derivatives exist. If it is possible to evaluate the integrand at unequally-spaced points, then other methods such as Gaussian quadrature and Clenshaw–Curtis quadrature are generally more accurate.

The zeroeth extrapolation, R(n, 0), is equivalent to the trapezoidal rule with 2n + 1 points; the first extrapolation, R(n, 1), is equivalent to Simpson's rule with 2n + 1 points. The second extrapolation, R(n, 2), is equivalent to Boole's rule with 2n + 1 points. Further extrapolations differ from Newton Cote's Formulas. In particular further Romberg extrapolations expand on Boole's rule in very slight ways, modifying weights into ratios similar as in Boole's rule. In contrast, further Newton Cotes methods produce increasingly differing weights, eventually leading to large positive and negative weights. This is indicative of how large degree interpolating polynomial Newton Cotes methods fail to converge for many integrals, while Romberg integration is more stable.

When function evaluations are expensive, it may be preferable to replace the polynomial interpolation of Richardson with the rational interpolation proposed by Bulirsch & Stoer.

What follows is an example of how Romberg method is implemented, the Gaussian function is integrated from 0 to 1, i.e. the error function erf(1) ≈ 0.842700792949715. The triangular array is calculated row by row and calculation is terminated if the two last entries in the last row differ less than 10^{-8}.

0.77174333
0.82526296 0.84310283
0.83836778 0.84273605 0.84271160
0.84161922 0.84270304 0.84270083 0.84270066
0.84243051 0.84270093 0.84270079 0.84270079 0.84270079

The result in the lower right corner of the triangular array is accurate to the digits shown. It is remarkable that this result is derived from the less accurate approximations obtained by the trapezium rule in the first column of the triangular array.

Second, in numerical analysis, we have many ways to approximate π . One approach is through the integral

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}.$$

We can write this integral as

$$\pi = 4\int_0^1 \frac{1}{1+x^2}.$$

Therefore the problem boils down to approximate the integral $\int_0^1 \frac{1}{1+x^2}$ numerically. The more accurate numerical approximation of $\int_0^1 \frac{1}{1+x^2}$ we get, the more accurate numerical approximation of π we get. Matlab provides a powerful tool for us to do this work.

What follows is the code for implement the approximation of π ;

```
function res=romberg(f,a,b,err)
m=1;k=1;
T=trapezoid(f,m,a,b);
for k=2:20
   m=2*m;
   s=trapezoid(f,m,a,b);
   T=[s,T];s=T(k);
   for p=1:k-1
      T(p+1)=(4^p*T(p)-T(p+1))/(4^p-1);
   end
   if abs(s-T(k)),err,break,end
end
s=[k,T(k)];
fprintf(1,[k,T(k)]).
```

The trapezoid function is defined as follows;

```
function res=trapezoid(f,n,a,b)
h=(b-a)/n;
res=feval(f,a)+feval(f,b);
x=a;
for k=1:n-1
   x=x+h;
   res=res+2*feval(f,x);
end
res=res*h/2;
```

```
Run
f=inline('4/(1+x^2)','x');
Romberg(f,0,1,.5e-5);
we get
3.14159265363824.
Run
f=inline('4/(1+x^2)','x');
Romberg(f,0,1,.5e-8);
we get
3.141592653589793.
```

This case means Matlab can be very helpful to the numerical approximation .

IV. SERIES CONVERGENCE ANALYSIS BY USING SIMBOLIC COMPUTING.

To understand this problem well, we first recall some fundamentals of series theory. A series is the sum of the terms of a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely.

In mathematics, given an infinite sequence of numbers $\{a_n, n = 1,2,...\}$, a series is informally the result of adding all those terms together: a1 + a2 + a3 + · · ·. These can be written more compactly using the summation symbol $\sum_{n=1}^{\infty} a_n$.

The terms of the series are often produced according to a certain rule, such as by a formula, or by an algorithm. As there are an infinite number of terms, this notion is often called an infinite series. Unlike finite summations, infinite series need tools from mathematical analysis to be fully understood and manipulated. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics and computer science.

There are a number of methods of determining whether a series converges or diverges. We list follows;

If the bigger series $\sum_{n=1}^{\infty} b_n$, can be proven to converge, then the smaller series $\sum_{n=1}^{\infty} a_n$ must converge. By contraposition, if the smaller series $\sum_{n=1}^{\infty} a_n$ is proven to diverge, then $\sum_{n=1}^{\infty} b_n$ must also diverge. That is the so called comparison test in which terms of the sequence are compared to those of another sequence.

Ratio test. Assume that for all n, $a_n > 0$. Suppose that there exists r such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = r$ .If r < 1, then the series converges. If r > 1, then the series diverges. If r = 1, the ratio test is inconclusive, and the series may converge or diverge.

Root test or nth root test. Suppose that the terms of the sequence in question are non-negative. Define r as follows: $\lim_{n\to\infty}\sqrt[n]{a_n} = r$.If r < 1, then the series converges. If r > 1, then the series diverges. If r = 1, the root test is inconclusive, and the series may converge or diverge.

The ratio test and the root test are both based on comparison with a geometric series, and as such they work in similar situations. In fact, if the ratio test works (meaning that the limit exists and is not equal to 1) then so does the root test; the converse, however, is not true. The root test is therefore more generally applicable, but as a practical matter the limit is often difficult to compute for commonly seen types of series.

Integral test. The series can be compared to an integral to establish convergence or divergence. Let $f(n)=a_n$ can be a positive and monotone decreasing function. If $\int_1^{+\infty} f(x)dx$ converges, then the series converges. But if the integral diverges, then the series does so as well.

Alternating series test. Also known as the Leibniz criterion, the alternating series test states that for an alternating series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$, if a_n is monotone decreasing, and has a limit of 0 at infinity, then the series converges.

If the series converges, then the series is absolutely convergent. An absolutely convergent sequence is one in which the length of the line created by joining together all of the increments to the partial sum is finitely long. The power series of the exponential function is absolutely convergent everywhere.

The Riemann series theorem states that if a series converges conditionally, it is possible to rearrange the terms of the series in such a way that the series converges to any value, or even diverges.

The above theory point out some tests to judge weather a series converges or diverges in a theoretical way. But in most cases it's a dubious job to implement these tests simply because in most cases its no easy to calculate the limit. So we may make use the symbolic function (see [13]) of mathematical software like Matlab to do this work. We give some examples as follows.

Example1： $\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$

Matlab Ratio Test： $\frac{a_{n+1}}{a_n}=2(1+\frac{1}{n})^n$

Syms n;
limit(2*(1+n)^(1/n),n,0)
ans =2*exp(1)>1;
So the series diverges .

Example2： $\sum_{n=1}^{\infty} n^2 e^{-n}$

Matlab Root Test：

$\sqrt[n]{a_n}=\sqrt[n]{n^2 e^{-n}}$

Syms n;
limit(((1/n)^2*(1/exp(n)))^(1/n),n,0)
syms n;
>> limit(((1/n)^2*(1/exp(1/n)))^n,n,0)
ans =
exp(-1)<1
So the series converges.

Example3： $\sum_{n=2}^{\infty}\frac{1}{n\ln^3 n}$

Matlab Integral Test：
>> int(1/(x*log(x)^3),x)
ans =
-1/2/log(x)^2
>> limit(-1/2/log(1/x)^2,x,0)
ans =0
Therefore, the series converges.

V. CONCLUSION

This paper presents several cases of Matlab applications in mathematical analysis including generating the graph of Weierstarass function, approximating circumference ratio and testing the convergence of series by symbolic computing. From these case studies, we can see Matlab can be a significant tool in mathematical analysis.

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