The Generalization of WOSF and Corresponding Outputs

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Abstract—**Weighted order statistic filters (WOSF) generate linearly separable Boolean functions. In some cases, 2 different WOSF may generate the same Boolean function. We thus can collect all of the WOSF together, which generate the same Boolean function. In this paper, we construct equivalent classes of WOSF, the BF equivalent class and the global equivalent class. Besides, we use minimum weight vectors as base vectors to build all global classes of same order and characterize WOSF to generalize three properties of global classes. Finally, we propose 3 translated formulas to fast generate corresponding outputs of BF equivalent classes, not perform extra machine training or mathematical computation.**

Index Terms— **hyperplane, WOSF, BF equivalent class, global equivalent class, base weight vector, corresponding outputs.**

I. INTRODUCTION

Weighted order statistic filters (WOSF) have a variety of applications such as noise cancellation, image reconstruction, edge enhancement, and texture analysis on Ref. [1-2]. WOSF have a large of variations according to the practical implementation. Some scholars have presented several papers about the representation of WOSF. Their representations of WOSF are based on the pair of a weight vector and a threshold value on Ref. [3-6].

Ref. [7-8] support vector machine (SVM) has the characterization of maximal margin classification, it is a well suited method for studying the problem of linearly separable Boolean functions. Ref. [9] Yih-Lon Lin, et al. makes use of SVM to implement linearly separable Boolean functions, Ref. [10] C. C. Yao and P. T. Yu use dichotomous approach to design weighted order statistic filters by SVM. Ref. [11] W. C. Chen and J. H. Jeng utilize SVM to represent WOS filters

In this paper, we propose a distinct method to character WOSF. The representation consists of two equivalent classes which are BF equivalent class and global equivalent class. The BF class is generated based on maximal margin classification by SVM. From SVM training, we generate a maximal margin hyperplane formulated by an optimal normal vector and an optimal bias. The hyperplane defines a discriminant function and this function has the same outputs as those of the WOSF. The normal vector and bias is unique, therefore they are used as the unique representative of the BF class.

The global class is constructed by sign change and permutation operations on the components of the representative of BF class. It is known that sign change and permutation operations on components of the optimal normal vector or the sign change on the optimal bias result in another pair of optimal parameters. Based on this concept, we can rearrange a given BF representative such that all the components of the parameter are positive and decreasing component-wise. This rearranged parameter also is a unique representative of the global class. Although the sign change and permutation operations can generate a new vector and a bias to build a BF equivalent class, but the corresponding outputs of BF class be not generated. Therefore, we proposed 3 translated formulas to fast generate these outputs of BF equivalent classes, not perform extra machine training or mathematical computation.

Besides, we collect the minimum weight vectors of WOSF in the order 2 to order 5 through SVM training. These weight vectors can be used to express all of global equivalent classes on same order, so we called them as base weight vectors. Finally, we characterize global equivalent classes by base weight vectors and generalize three properties. These properties explicitly describe the characterization of equivalent classes and show the computation of the numbers of global equivalent classes.

In other words, we can efficiently represent all of the WOSF through only few representatives of equivalent classes and save computation cost when searching for various WOSF.

II. EQUIVALENT CLASSES OF WOSF

For a given WOSF (Ω_0, r_0), $[(\Omega_0, r_0)]_{BF}$ denote the equivalent class such that all the WOSF in this set generates the same linearly separable Boolean function.

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For these WOSF, we can find one that is an unique maximal margin hyperplane (w^*, b^*) through SVM. Therefore we can regard the term (w^*, b^*) as the unique representative of the class $[(\Omega_0, r_0)]_{BF}$.

Let WOSF $(\Omega^*, r^*) \in [(\Omega_0, r_0)]_{BF}$, in order to compromise the notation, we define a new function $F_{\Omega^*, r^*}(x): \{-1, 1\}^n \to \{-1, 1\}$. This function $F_{\Omega^*, r^*}(x)$ is linearly separable, so it can be rewritten to

$$F_{\Omega^*, r^*}(x) = \begin{cases} +1 & \text{if } <\Omega^*, x> + r^* \geq 0 \\ -1 & \text{else} \end{cases} \quad (1)$$

In terms of discriminant function, this function can be formed as

$$\operatorname{sgn}(F_{\Omega^*, r^*}(x)) = \operatorname{sgn}(<\Omega^*, x> + r^*) \quad (2)$$

The sign of the discriminant function obtained from SVM

$$\operatorname{sgn}(f_{w^*, b^*}(x)) = \operatorname{sgn}(<w^*, x> + b^*) \quad (3)$$

also generates the same Boolean function with equation (2). As a consequence, we can regard the pair (w^*, b^*) as the unique representative of the equivalent class $[(\Omega_0, r_0)]_{BF}$ and denoted as $[(w^*, b^*)]_{BF}$.

Under the concept of equivalent class and the property of unique representative through SVM, we will characterize the set of all WOSF.

To explore other distinct equivalent classes without performing the same SVM procedure above, we operate on this representative by simple sign change and permutation on the quantities w^* and b^*, respectively. Without the SVM training, we can therefore save the cost of complicated SVM training process.

For fixed b, if 2 components of w are swapped or the sign of one component of w is changed, the vector w can generate a new vector, denoted by $\hat{w}$, we will therefore obtain a new equivalent class of WOSF. The new vector $\hat{w}$ is the unique representative of new class in the sense of SVM classification, which has the margin $\| \hat{w} \|^{-1} = \| w \|^{-1}$. In addition, when w fixed, the sign of b is altered, denoted as $\hat{b}$, we will also obtain another new equivalent class. Each new equivalent class is a new hyperplane, these new hyperplanes all have the same margin $\| w \|^{-1}$. This suggests another equivalent relation.

Given a maximal margin hyperplane (w, b), we define the global equivalent class $[(w, b)]_G$ as the set of equivalent classes $[(w, b)]_{BF}$ such that if $\hat{w}$ is a permutation or sign change of w, or $\hat{b}$ is the sign change of b then $[(\hat{w}, \hat{b})]_{BF} \in [(w, b)]_G$. Here, we choose the symbol $(w^\#, b^\#)$ in which $w^\# = (w_1^\#, ..., w_n^\#)$, $w_1^\# \geq w_2^\# \geq ... \geq w_n^\# \geq 0$, and $b^\# \geq 0$ as the representative of the global equivalent class and this global equivalent class $[(w, b)]_G$ is then denoted by $[(w^\#, b^\#)]_G$. It is easy seen that the representative is unique. It should be noted that each element in the global equivalent class is itself a BF equivalent class.

For example, let $w^\# = (2\ 1\ 1)$ and $b^\# = 1$, if they are generated by SVM training based on the property of maximal margin classification, then the pair $(w^\#, b^\#)$ is the unique representative of BF equivalent class $[(w^\#, b^\#)]_{BF}$. Apply permutation and sign operations on $w^\#$, it yields additional 23 new vectors $(1\ 2\ 1)$, $(1\ 1\ 2)$,…, and $(1 -1\ 2)$. Similarly, when sign change on $b^\#$ also yields additional 24 combinations. Therefore, including the origin one $w^\#$ and $b^\#$, it adds up to 48 hyperplanes and builds 48 BF equivalent classes. These BF classes form a global equivalent class $[(w^\#, b^\#)]_G$ and $w^\#$ and $b^\#$ is the unique representative of the global class. Table II lists all of these hyperplanes. And as mentioned above, all of the 48 hyperplanes have the same maximum margin $1/\sqrt{6}$.

III. OUTPUTS GENERATION

In above section, we have utilized the sign change and permutation of base weight vector w and bias b to generate many new vectors and bias. These new vectors and bias group into some BF equivalent classes. Each BF equivalent class has a distinct corresponding output. But, the output does not be generated by sign change or permutation. Therefore, we further propose 3 translated formulas to generate these outputs. Let (w_1, b_1) and (w_2, b_2) are respectively the representatives of BF equivalent classes $[(w_1, b_1)]_{BF}$ and $[(w_2, b_2)]_{BF}$ derived from two distinct maximal margin hyperplanes. The BF class $[(w_1, b_1)]_{BF}$ has the corresponding output y_1 and the BF class $[(w_2, b_2)]_{BF}$ has the corresponding output y_2. Based on these formulas, we can directly generate the corresponding outputs of BF equivalent classes on same order, not need to perform extra training or mathematical computation. These formulas are listed in the following propositions:

A. *Proposition 1—weight vector sign change*

For $b_2 = b_1$, let w_2 is the new vector of 1-component sign change of w_1, i.e.,

$w_1 = [w_{11}, ..., w_{1j}, ..., w_{1n}]^T$ and $w_2 = [w_{11}, ..., -w_{1j}, ..., w_{1n}]^T$, then the outputs y_2 can be computed by the outputs y_1

$$y_2 = P^{2^{(n-j)}} y_u + P^{-2^{(n-j)}} y_v \quad (4)$$

where $y_1 = y_u + y_v$, $j = 1...n$, and P is the $n \times n$ matrix

$$P = \begin{bmatrix} 0 & 1 & . & . & . & 0 \\ . & . & 1 & . & . & . \\ . & . & . & \ddots & . & . \\ . & . & . & . & \ddots & . \\ 0 & . & . & . & . & 1 \\ 1 & 0 & . & . & . & 0 \end{bmatrix}_{n\times n} .$$

The terms y_u and y_v are defined as

$$y_u = \sum_{m=1}^{2^{j-1}} \sum_{u=0}^{2^{n-j}-1} (\tilde{y}_{1(2^{(n-j+1)}m-u)}) ,$$

$$y_v = \sum_{m=1}^{2^{j-1}} \sum_{v=2^{n-j}}^{2^{n-j+1}-1} (\tilde{y}_{1(2^{(n-j+1)}m-v)}) ,$$

where $\tilde{y}_{1j} = [0 ... y_{1j} ... 0]^T$.

B. *Proposition 2—weight vector permutation*

For $b_2 = b_1$, let w_2 is the new vector of 2-components permutation of w_1, i.e., $w_1 = [w_{11} .... w_{1(j-1)}\ w_{1j} ... w_{1n}]^T$ and $w_2 = [w_{11} ... w_{1j}\ w_{1(j-1)} ... w_{1n}]^T$, then the outputs y_2 can be computed by the outputs y_1

$$y_2 = P^{2^{(n-j)}} y_u + P^{-2^{(n-j)}} y_v + \sum_{j \notin u,v} \tilde{y}_{1j} \quad (5)$$

where $y_1 = y_u + y_v$, $j = 1...n$, and P is the $n \times n$ matrix

$$P = \begin{bmatrix} 0 & 1 & . & . & . & 0 \\ . & . & 1 & . & . & . \\ . & . & . & \ddots & . & . \\ . & . & . & . & \ddots & . \\ 0 & . & . & . & . & 1 \\ 1 & 0 & . & . & . & 0 \end{bmatrix}_{n\times n}$$

The terms y_u and y_v are defined as

$$y_u = \sum_{m=odd}^{2^{j-1}-1} \sum_{u=-2^{n-j}}^{-1} (\tilde{y}_{1(2^{(n-j+1)}m-u)})$$

$$y_v = \sum_{m=odd}^{2^{j-1}-1} \sum_{v=1-2^{n-j}}^{0} (\tilde{y}_{1(2^{(n-j+1)}m+v)})$$

where $\tilde{y}_{1j} = [0 ... y_{1j} ... 0]^T$.

C. *Proposition 3— bias sign change*

For $w_2 = w_1$, let b_2 is the new bias of sign change of b_1, i.e., $b_2 = -b_1$, then the outputs y_2 can be computed by the outputs y_1

$$y_2 = NOT(M\ y_1) \quad (6)$$

where M is an $n \times n$ matrix of the form

$$M = \begin{bmatrix} 0 & 0 & . & . & 0 & 1 \\ 0 & . & . & . & 1 & 0 \\ . & . & . & \iddots & . & . \\ . & . & \iddots & . & . & . \\ 0 & 1 & . & . & . & . \\ 1 & 0 & . & . & . & 0 \end{bmatrix}_{n\times n}$$

IV. PROPERTIES OF GLOBAL EQUIVALENT CLASS

For any global class $[(w^\#, b^\#)]_G$, $w^\# = (w_1^\#, ..., w_n^\#)$, $w_1^\# \geq w_2^\# \geq ... \geq w_n^\# \geq 0$, and $b^\# \geq 0$. The Hamming weight of $w^\#$ is $|w^\#| = r$. Vector $w^\#$ can generate different global equivalent classes $[(w', b')]_G$, and the vector w' has less or equal order with $w^\#$, where $w' = (w'_1, ..., w'_n)$, $b' \leq r - 1$. Based on the characterization of maximal margin, one has following properties:

A. *Property 1*

For arbitrary global equivalent class $[(w^\#, b^\#)]_G$, $w^\# = (w_1^\#, ..., w_n^\#)$, $w_1^\# \geq w_2^\# \geq ... \geq w_n^\# \geq 0$, $b^\# \geq 0$. When $w_i^\# \neq w_j^\#$, $1 \leq i, j \leq n$, the global class contains $2^{n+1} \cdot n!$ BF equivalent classes.

For this class $[(w^\#, b^\#)]_G$, all elements are generated by permutation or sign change of $w^\#$ and sign change of $b^\#$. Let elements $(\hat{w}, \hat{b}) \in [(w^\#, b^\#)]_G$, each element $(\hat{w}, \hat{b})$ represent a BF equivalent class. When $w_i^\# \neq w_j^\#$, by permutation, the vector $w^\#$ generates $n!$ different weight vectors w'. By sign change, each generated vector w' can regenerate 2^n different weight vectors $\hat{w}$. So, the vector $w^\#$ can generate $2^n \times n!$ different vectors in all. Besides, by the sign change of $b^\#$, it also generates two thresholds, $\hat{b}$ and $-\hat{b}$. Each $\hat{w}$

integrates $\hat{b}$ to build a BF equivalent class, therefore the permutation and sign change of $w^{\#}$ and the sign change of $b^{\#}$ that can together build $2^{n+1} \cdot n!$ BF equivalent classes. For special case, when vector $w^{\#}$ has q same components, implies $w_i^{\#} = w_j^{\#} = ... = w_m^{\#}$, by permutation, the vector $w^{\#}$ can generates $\frac{n!}{q!}$ different weight vectors w'. Hence, the $w^{\#}$ and $b^{\#}$ together build $2^{n+1} \cdot \frac{n!}{q!}$ BF equivalent classes.

Example, for global class $[(43211), 2]_G$, the weight vector is $w^{\#} = (43211)$. By permutation, vector (43211) generates 60 different weight vectors w', by sign change, each vector w' regenerates 32 new vector $\hat{w}$. So, the permutation or sign change of vector (43211) and the sign change of threshold $b^{\#} = 2$ that can together build 3840 BF equivalent classes.

B. *Property 2*

For arbitrary global class $[(w^{\#}, b^{\#})]_G$, the Hamming weight $|w^{\#}| = r$. When r is odd, the $w^{\#}$ generates $\frac{r+1}{2}$ different global equivalent classes. When r is even, the $w^{\#}$ generates $\frac{r}{2}$ different global equivalent classes.

For this class $[(w^{\#}, b^{\#})]_G$, the element $(w^{\#}, b^{\#})$ represents a BF equivalent class. This class is composed of some WOSF with the same Boolean function. Let the function $f_{w^{\#},b^{\#}}(x)$ represent a WOSF,

$$f_{w^{\#},b^{\#}}(x) = \begin{cases} 1, & < w^{\#}, x > \geq b^{\#} \\ -1, & \text{else} \end{cases}$$

The outputs of $f_{w^{\#},b^{\#}}(x)$ are based on the computation between product $< w^{\#}, x >$ and thresholds $b^{\#}$. Since the input is $x \in \{-1, 1\}$, for each threshold b' and interval $[t-2, t]$. When $b' \in [t-2, t]$, the WOSF $f_{w^{\#},b'}(x)$ generates same Boolean functions.

When r is odd, based on above description, the weight r could be divided into $\frac{r+1}{2}$ intervals, $[-1, 1]$, $[1, 3]$, ……, and $[r-2, r]$. When r is even, it be divided into $\frac{r}{2}$ intervals, $[0, 2]$, $[2, 4]$, ……, and $[r-2, r]$. For each interval, one can find a threshold b' such that the threshold b' integrates the vector $w^{\#}$ to form a hyperplane $(w^{\#}, b')$, this hyperplane can be used to represent a BF equivalent class. In other words, when r is odd, the vector $w^{\#}$ generates $\frac{r+1}{2}$ different global equivalent classes, and when r is even, the vector $w^{\#}$ generates $\frac{r}{2}$ different global equivalent classes.

For the class $[(52211), 2]_G$, the Hamming weight $|w^{\#}| = 11$. Weight 11 is divided into 6 intervals, $[-1, 1]$, $[1, 3]$, $[3, 5]$, $[5, 7]$, $[7, 9]$ and $[9, 11]$. So the vector (52211) can generate 6 different classes $[(31111), 0]_G$, $[(52211), 2]_G$, $[((42211), 3)]_G$, $[((32211), 4)]_G$, $[((22211), 5)]_G$ and $[((11111), 4)]_G$. For this class $[((53211), 1)]_G$, the Hamming weight $|w^{\#}| = 12$. Weight 12 is divided into 6 intervals, $[0, 2]$, $[2, 4]$, $[4, 6]$, $[6, 8]$, $[8, 10]$, and $[10, 12]$. So the vector (53211) generates 6 different classes $[((42211), 1)]_G$, $[((53211), 3)]_G$, $[((32111), 3)]_G$, $[((33211), 5)]_G$, $[((11110), 3)]_G$, and $[((11111), 4)]_G$.

All vectors w' in the generated global classes that may have less or equal order with known vectors $w^{\#}$.For this class $[((53211), 1)]_G$, the vector (53211) generates 6 classes, and 5 vectors in these classes have the same order with $w^{\#}$.Another class $[((11110), 3)]_G$, the vector (11110) only has order 4.

C. *Property 3*

For arbitrary two global equivalent classes of the same order $[(w^{\#}, b^{\#})]_G$ and $[(w', b')]_G$, $w' \neq w^{\#}$. The vector $w^{\#}$ has Hamming weight $|w^{\#}| = r$ and the vector w' has Hamming weight $|w'| = r'$, then the BF equivalent classes $[(w^{\#}, r-1)]_{BF}$ and $[(w', r'-1)]_{BF}$ are the same.

Let the functions $f_{w^{\#},b^{\#}}(x)$ and $f_{w',b'}(x)$ respectively represent two distinct WOSF. Since the input $x \in \{-1, 1\}$, each value of inner product $< w', x >$ is shown by unit 2. By property 2, when thresholds $b' \in [r'-2,\ r']$, the WOSF $f_{w',b'}(x)$ generates some Boolean functions and these Boolean functions have the same outputs $(-1, -1, ..., -1, 1)$. Similarly, when thresholds $b^{\#} \in [r-2,\ r]$, the WOSF $f_{w^{\#},b^{\#}}(x)$ also generates some Boolean functions, these Boolean functions also have the same outputs $(-1, -1, ..., -1, 1)$. Briefly, the WOSF (w', b') and $(w^{\#}, b^{\#})$ generate a group of same Boolean functions. In other words, the BF equivalent classes are the same, $[(w', b')]_{BF} = [(w^{\#}, b^{\#})]_{BF}$. When $b^{\#} = r-1$ and

$b'=r'-1$, the $[(w^{\#}, r-1)]_{BF} = [(w', r'-1)]_{BF}$.

For $n=5$, given two global equivalent classes $[((43221),1)]_G$ and $[((54321),2)]_G$. The vectors $w=(43221)$ with Hamming weight $|w|=12$ and vector $w'=(54321)$ with Hamming weight $|w'|=15$. The BF equivalent classes $[((43221),11)]_{BF}$ and $[((54321),14)]_{BF}$ generate the same Boolean function, hence $[((43221),11)]_{BF} = [((54321),14)]_{BF}$.

V. EXPERIMENT AND ILLUSTRATION

One takes an arbitrary WOSF $F_{(131),2}(x)$ that has a pack of output $(-1\,-1\;1\;1\,-1\;1\;1\;1)$. Composing the all possible inputs $x \in \{-1,1\}^3$ and corresponding outputs $y_i \in \{-1,1\}$ to form a training set $S = \{(x_i, y_i)\}_{i=1}^{8}$. From SVM training, we obtain a hyperplane with normal vector $w^* = (1\;2\;1)$ and bias $b^* = 1$. In order to the convenient vision, we take the symbol of "0" instead of "-1", therefore the outputs $(-1\,-1\;1\;1\,-1\;1\;1\;1)$ are rewritten down as $(0\,0\,1\,1\,0\,1\,1\,1)$.

Because the WOSF $((1\;2\;1),1)$ is characterized by the output $(0\,0\,1\,1\,0\,1\,1\,1)$, the signed discriminant function $\text{sgn}(f_{(121),1}(x))$ can represent all WOSF with the same outputs but different weight vectors and thresholds. Therefore, one can use a BF equivalent class $[(1\;2\;1),1)]_{BF}$ to present these WOSF with the same Boolean function. The relationship among WOSF, hyperplane and BF equivalent class are listed in the second column of Table I.

From the second column in Table I, the WOSF $F_{(131),2}(x)$, $F_{(231),3}(x)$, $F_{(243),4}(x)$, $F_{(154),5}(x)$ and $F_{(364),6}(x)$ all have the same outputs $(0\,0\,1\,1\,0\,1\,1\,1)$. According to Section II, we can construct the BF equivalent class $[(1\;2\;1),1]_{BF}$ in which the pair $((1\;2\;1),1)$ is the unique representative. Note that there are infinitely many elements in the class $[(1\;2\;1),1]_{BF}$ and here only 5 instances are listed in the column. The other columns in Table I show additional examples of BF equivalent classes which is listed in the third row.

In Table II, we show the all elements of global class $[(2\;1\;1),1]_G$ which contains 48 BF equivalent classes. Each element can be obtained from vector $w=(2\;1\;1)$ and bias $b=1$ by sign change and permutation. For example, the representative $((1\;2\,-1),-1)$ of a BF equivalent class is obtained by permutation on the first and the second components, and sign changes on the third component and the bias.

Also, we search the minimum collection of vectors in the same order through SVM training. In table III, we list the minimum collection of vectors from order 2 to order 5. The forth row shows the collected 30 base vectors and built 55 global classes of order 5. For order 4, we collect 6 base weight vectors, and these vectors build 10 global equivalent classes listed in table IV.

From table IV, we take the global class $[(3\,2\,1\,1),2]_G$ as an example to illustrate the outputs generation. In table V, we show all the vectors of permutation by vector $(3\,2\,1\,1)$ and corresponding outputs on left side, and show all the vectors of sign change and corresponding outputs on right side. For example, when fixed $b=2$, the base vector $(3\,2\,1\,1)$ of global equivalent class $[(3\,2\,1\,1),2]_G$ has corresponding output $y=(00\,0\,0\,0\,0\,0\,0\;\;0\,0\,0\,1\,1\,1\,1\,1)$. On the 1rd row of right side, we show the new vector $\hat{w}=(3\,2\,1\,-1)$ of sign change and corresponding output $\hat{y}=(00\,0\,0\,0\,0\,0\,0\;\;0\,0\,1\,0\,1\,1\,1\,1)$ generated by equation (4). On the 1rd row of left side we show the new vector $\hat{w}=(3121)$ of permutation and corresponding output $\hat{y}=(00\,0\,0\,0\,0\,0\,0\;\;0\,0\,1\,1\,0\,1\,1\,1)$ generated by equation (5). Other vectors and corresponding outputs are listed on other rows in table V.

One also takes all vectors of permutation from table V to build a new table, table VI. The table VI shows the fixed vector and corresponding output. For example, when sign change of $b=2$ into $b=-2$, the 1rd row shows the base vector $w=(3\,2\,1\,1)$, and corresponding output $\hat{y}=(00\,0\,0\,0\,1\,1\,1\;\;1\,1\,1\,1\,1\,1\,1\,1)$ generated by equation (6). The 2rd row shows the vector $w=(3\,1\,2\,1)$ of permutation, and corresponding output $\hat{y}=(00\,0\,1\,0\,0\,1\,1\;\;1\,1\,1\,1\,1\,1\,1\,1)$. The additional outputs are listed on other rows in Table VI.

VI. CONCLUSION

In this paper, we propose an alternative method to characterize WOSF. The characterization consists of two stages of equivalent classes. The first class, referred to as BF class, is generated based on maximal margin classification by SVM. From SVM training, we generate a maximal margin hyperplane formulated by an optimal normal vector and an optimal bias. The normal vector and bias are unique, Therefore they are used as the unique representative of the BF class. The second class, referred to as global class, is constructed by sign change and permutation operations on the components of the representative of BF class. The underlying idea is that sign change and permutation operations on components of the optimal normal vector and the optimal bias result in another pair of optimal parameters. This pair of parameters formulates another maximal margin hyperplane separating another linearly separable Boolean function.

Although the sign change and permutation operations can generate new vector and bias to group distinct BF equivalent classes, but the corresponding output of each BF equivalent class is not directly find. Therefore, we proposed 3 translated formulas to fast generate these outputs of BF equivalent classes, not perform extra training or build the truth table.

Besides, we collect minimum weight vectors as base vectors of WOSF in the same order. These base vectors can be used to generate all of global equivalent classes. Also, we characterize all of global equivalent classes by base vectors and propose three properties. These properties explicitly explain the character of equivalent classes, and can compute the numbers of equivalent classes. Therefore we can efficiently represent all of the WOSF through only few representatives of equivalent classes and save computation cost when search for various WOSF.

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TABLE I.
THE WOSF REPRESENTATION AND BF EQUIVALENT CLASSES CORRESPONDING TO THE SAME OUTPUTS

Output	(01111111)	(00110111)	(00110011)	(00010011)
Hyperplane	((111),2) sgn($f_{(111),2}$)	((121),1) sgn($f_{(121),1}$)	((101),0) sgn($f_{(010),0}$)	((121),-1) sgn($f_{(121),-1}$)
BF class $[w, b]_{BF}$	$[(1\ 1\ 1),2]_{BF}$	$[(1\ 2\ 1),1]_{BF}$	$[(1\ 0\ 1),0]_{BF}$	$[(1\ 2\ 1),-1]_{BF}$
WOSF (Ω, r)	((1 3 1),1)	((1 3 1),2)	((1 3 1),3)	((1 3 1),4)
WOSF (Ω, r)	((3 2 4),2)	((2 3 1),3)	((1 2 0),2)	((1 2 1),3)
WOSF (Ω, r)	((5 4 3),3)	((2 4 3),4)	((2 4 1),4)	((2 4 3),6)
WOSF (Ω, r)	((4 7 4),4)	((1 5 4),5)	((2 5 2),5)	((2 5 2),7)
WOSF (Ω, r)	((5 8 9),5)	((3 6 4),6)	((2 6 3),6)	((4 6 5),10)

TABLE II.
THE ALL ELEMENTS OF GLOBAL CLASS $[(2\ 1\ 1),1]_G$

$[\hat{w}, b]_{BF}$			$[\hat{w}, \hat{b}]_{BF}$		
$\hat{w}$	b	outputs	$\hat{w}$	$\hat{b}$	outputs
(2 1 1)	1	(0 0 0 1 1 1 1 1)	(2 1 1)	-1	(0 0 0 0 0 1 1 1)
(2 1-1)	1	(0 0 1 0 1 1 1 1)	(2 1-1)	-1	(0 0 0 0 1 0 1 1)
(1 2 1)	1	(0 0 1 1 0 1 1 1)	(2-1 1)	-1	(0 0 0 0 1 1 0 1)
(1 2-1)	1	(0 0 1 1 1 0 1 1)	(2-1-1)	-1	(0 0 0 0 1 1 1 0)
(2-1 1)	1	(0 1 0 0 1 1 1 1)	(1 2 1)	-1	(0 0 0 1 0 0 1 1)
(1 1 2)	1	(0 1 0 1 0 1 1 1)	(1 1 2)	-1	(0 0 0 1 0 1 0 1)
(1-1 2)	1	(0 1 0 1 1 1 0 1)	(1 2-1)	-1	(0 0 1 0 0 0 1 1)
(-1 2 1)	1	(0 1 1 1 0 0 1 1)	(1 1-2)	-1	(0 0 1 0 1 0 1 0)
(-1 1 2)	1	(0 1 1 1 0 1 0 1)	(-1 2 1)	-1	(0 0 1 1 0 0 0 1)
(2-1-1)	1	(1 0 0 0 1 1 1 1)	(-1 2-1)	-1	(0 0 1 1 0 0 1 0)
(1 1-2)	1	(1 0 1 0 1 0 1 1)	(1-1 2)	-1	(0 1 0 0 0 1 0 1)
(1-1-2)	1	(1 0 1 0 1 1 1 0)	(1-2 1)	-1	(0 1 0 0 1 1 0 0)
(-1 2-1)	1	(1 0 1 1 0 0 1 1)	(-1 1 2)	-1	(0 1 0 1 0 0 0 1)
(-1 1-2)	1	(1 0 1 1 1 0 1 0)	(-1-1 2)	-1	(0 1 0 1 0 1 0 0)
(1-2 1)	1	(1 1 0 0 1 1 0 1)	(-2 1 1)	-1	(0 1 1 1 0 0 0 0)
(1-2-1)	1	(1 1 0 0 1 1 1 0)	(1-1-2)	-1	(1 0 0 0 1 0 1 0)
(-1-1 2)	1	(1 1 0 1 0 1 0 1)	(1-2-1)	-1	(1 0 0 0 1 1 0 0)
(-1-2 1)	1	(1 1 0 1 1 1 0 0)	(-1 1-2)	-1	(1 0 1 0 0 0 1 0)
(-1-1-2)	1	(1 1 1 0 1 0 1 0)	(-1-1-2)	-1	(1 0 1 0 1 0 0 0)
(-1-2-1)	1	(1 1 1 0 1 1 0 0)	(-2 1-1)	-1	(1 0 1 1 0 0 0 0)
(-2 1 1)	1	(1 1 1 1 0 0 0 1)	(-1-2 1)	-1	(1 1 0 0 0 1 0 0)
(-2 1-1)	1	(1 1 1 1 0 0 1 0)	(-1-2-1)	-1	(1 1 0 0 1 0 0 0)
(-2-1 1)	1	(1 1 1 1 0 1 0 0)	(-2-1 1)	-1	(1 1 0 1 0 0 0 0)
(-2-1-1)	1	(1 1 1 1 1 0 0 0)	(-2-1-1)	-1	(1 1 1 0 0 0 0 0)

TABLE III.
THE BASE WEIGHT VECTORS FROM ORDER 2 TO ORDER 5

Order	Vector Num.	Global class	Base weight vector
2	1	1	(1 1)
3	2	3	(1 1 1)(2 1 1)
4	6	10	(1 1 1 1)(2 1 1 1)(2 2 1 1)(3 1 1 1) (3 2 1 1)(3 2 2 1)
5	30	55	(1 1 1 1 1)(2 1 1 1 1)(3 1 1 1 1)(4 1 1 1 1) (2 2 1 1 1)(3 2 1 1 1)(3 3 1 1 1)(4 2 1 1 1) (4 3 1 1 1)(2 2 2 1 1)(3 2 2 1 1)(3 3 2 1 1) (4 2 2 1 1)(4 3 2 1 1)(4 3 3 1 1)(5 2 2 1 1) (5 3 2 1 1)(5 3 3 1 1)(3 2 2 2 1)(3 3 2 2 1) (4 3 2 2 1)(4 3 3 2 1)(5 2 2 2 1)(5 3 2 2 1) (5 3 3 2 1)(5 4 2 2 1)(5 4 3 2 1)(3 3 2 2 2) (4 3 3 2 2)(5 4 3 2 2)

TABLE IV.
THE ALL GLOBAL CLASSES BUILT BY BASE VECTORS ON ORDER 4

Num.	Base vector	Global classes
1	(1 1 1 1)	$[(1\ 1\ 1\ 1),1]_G$, $[(1\ 1\ 1\ 1),3]_G$
2	(2 1 1 1)	$[(2\ 1\ 1\ 1),0]_G$, $[(2\ 1\ 1\ 1),2]_G$
3	(2 2 1 1)	$[(2\ 2\ 1\ 1),1]_G$, $[(2\ 2\ 1\ 1),3]_G$
4	(3 1 1 1)	$[(3\ 1\ 1\ 1),1]_G$
5	(3 2 1 1)	$[(3\ 2\ 1\ 1),2]_G$
6	(3 2 2 1)	$[(3\ 2\ 2\ 1),1]_G$, $[(3\ 2\ 2\ 1),5]_G$

TABLE V.
THE GENERATED VECTORS AND CORRESPONDING OUTPUTS
WHEN SIGN CHANGE AND PERMUTATION OF BASE VECTOR (3 2 1 1)

Permutation		Sign Change	
w	Outputs	*w*	Outputs
(3 2 1 1)	(00000000 00011111)	(3 2 1 1)	(00000000 00011111)
(3 1 2 1)	(00000000 00110111)	(3 2 1-1)	(00000000 00101111)
(3 1 1 2)	(00000000 01010111)	(3 2-1 1)	(00000000 01001111)
(2 3 1 1)	(00000001 00001111)	(3-2 1 1)	(00000000 11110001)
(2 1 3 1)	(00000001 00110011)	(-3 2 1 1)	(00011111 00000000)
(2 1 1 3)	(00000001 01010101)	(3 2-1-1)	(00000000 10001111)
(1 3 2 1)	(00000011 00000111)	(3-2 1-1)	(00000000 11110010)
(1 3 1 2)	(00000101 00000111)	(3-2-1 1)	(00000000 11110100)
(1 2 3 1)	(00000011 00010011)	(-3 2 1-1)	(00101111 00000000)
(1 2 1 3)	(00000101 00010101)	(-3 2-1 1)	(01001111 00000000)
(1 1 3 2)	(00010001 00010011)	(-3-2 1 1)	(11110001 00000000)
(1 1 2 3)	(00010001 00010101)	(3-2-1-1)	(00000000 11111000)

TABLE VI.
THE GENERATED VECTORS BY VECTOR (3 2 1 1) AND CORRESPONDING OUTPUTS
WHEN SIGN CHANGE OF BIAS *b*

w	*b*	Outputs	*w*	*b*	Outputs
(3 2 1 1)	2	(00000000 00011111)	(3 2 1 1)	-2	(00000111 11111111)
(3 1 2 1)	2	(00000000 00110111)	(3 1 2 1)	-2	(00010011 11111111)
(3 1 1 2)	2	(00000000 01010111)	(3 1 1 2)	-2	(00010101 11111111)
(2 3 1 1)	2	(00000001 00001111)	(2 3 1 1)	-2	(00001111 01111111)
(2 1 3 1)	2	(00000001 00110011)	(2 1 3 1)	-2	(00110011 01111111)
(2 1 1 3)	2	(00000001 01010101)	(2 1 1 3)	-2	(01010101 01111111)
(1 3 2 1)	2	(00000011 00000111)	(1 3 2 1)	-2	(00011111 00111111)
(1 3 1 2)	2	(00000101 00000111)	(1 3 1 2)	-2	(00011111 01011111)
(1 2 3 1)	2	(00000011 00010011)	(1 2 3 1)	-2	(00110111 00111111)
(1 2 1 3)	2	(00000101 00010101)	(1 2 1 3)	-2	(01010111 01011111)
(1 1 3 2)	2	(00010001 00010011)	(1 1 3 2)	-2	(00110111 01110111)
(1 1 2 3)	2	(00010001 00010101)	(1 1 2 3)	-2	(01010111 01110111)