Calculating Weights Methods in Complete Matrices and Incomplete Matrices

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***Abstract*—It is well known that the Analytic Hierarchy Process (AHP) of Saaty is one of the most powerful approach for decision aid in solving of a multi criteria decision making (MCDM) problem. Several computing weights methods in AHP are analyzed. Based on least square method, three methods for calculating weights using the least the sum of squares of error criterion, the least the sum of error absolute value criterion and the least the error absolute value criterion are proposed. New least squares method is translated into linear system and Minimax method and absolute deviation method are translated into linear programming. New proposed methods can apply to the ranking estimation in incomplete AHP, which is very important to estimate incomplete comparisons data to have alternative's weights. The computation methods and results are given through numerical examples. The new methods have fast convergence and smaller computational complexity.**

***Index Terms*— analytic hierarchy process (AHP) , weights, error, linear programming, incomplete matrices**

I. Introduction

The Analytic Hierarchy Process (AHP) is a multi-criteria decision-making approach and was introduced by Saaty (1977 and 1994). The AHP has attracted the interest of many researchers mainly due to the nice mathematical properties of the method and the fact that the required input data are rather easy to obtain. The AHP is a decision support tool which can be used to solve complex decision problems. It uses a multi-level hierarchical structure of objectives, criteria, subcriteria, and alternatives. The pertinent data are derived by using a set of pairwise comparisons. These comparisons are used to obtain the weights of importance of the decision criteria, and the relative performance measures of the alternatives in terms of each individual decision criterion. If the comparisons are not perfectly consistent, then it provides a mechanism for improving consistency. Some of the industrial engineering applications of the AHP include its use in integrated manufacturing (Putrus, 1991), in the evaluation of technology investment decisions (Boucher and McStravic, 1991), in flexible manufacturing systems (Wabalickis, 1988), layout design (Cambron and Evans, 1991), and also in other engineering problems (Wang and Raz, 1991). The most common techniques for an estimating relative priority weights is originally proposed eigenvector method. Recently, a many alternative approaches developed from the least square method to goal programming are found in the many numbers of references. Based on the least deviations priority method (LDM) given by Chen Baoqian (1990), Wang Yingming(1993) proposed a new class of generalized least deviations priority methods (GLDM) of comparison matrix in analytic hierarchy process and also gives a convergent iterative algorithm and a simulation example. Zhang Zhimin (1996 and 1997) discuss some properties of Least deviations method in AHP and investigated the basic properties of MLSM. Based on least square method, three methods for calculating weights using the least the sum of squares of error criterion, the least the sum of error absolute value criterion and the least the error absolute value criterion are proposed. New proposed methods can apply to the ranking estimation in incomplete AHP.

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II. Several usual calculating methods to AHP problem

There are numerous methodology presented in many publications for deriving priority weights in the AHP. Practically, the most common approach is the originally proposes eigenvector method.

A. Sum Method

Let $A=(a_{ij})$ a is n x n judgement matrix. Firstly we normalize the column vectors in the judging matrix, then add the normalized matrix in rows. The result should be normalized again to get the eigenvector:

$$w_i=\frac{1}{n}\sum_{j=1}^{n}\frac{a_{ij}}{\sum_{k=1}^{n}a_{kj}} \quad (i=1,2,\cdots,n) \qquad (1)$$

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B. Geometric Mean Method

The geometric mean method is defined by

$$w_i = \frac{(\prod_{j=1}^{n} a_{ij})^{\frac{1}{n}}}{\sum_{k=1}^{n}(\prod_{j=1}^{n} a_{kj})^{\frac{1}{n}}} \quad (i = 1, 2, \cdots, n) \qquad (2)$$

The geometric mean solution can be derived as the solution of following optimization problem:

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}[\ln a_{ij} - \ln(w_i / w_j)]^2$$

$$s.t. \quad \sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1,2,\cdots,n.$$

C. Eigenvector Method

It consists in taking as weights the components of the (right) eigenvector of the matrix A. In our notation the eigenvector is defined by

$$AW = \lambda_{\max} W \qquad (3)$$

Where $\lambda_{\max}$ is the largest eigenvalue of A. It must be noted that this eigenvector solution is normalized additively, i. e. $\sum_{i=1}^{n} w_i = 1$.

D. Least Square Method

Construct generalized deviations function

$$f(w_1, w_2, \cdots, w_n) = \sum_{1 \le i \le j \le n}[a_{ij} - w_i / w_j]^2 .$$

Obviously, the reasonable weight vector $W = (w_1, w_2, \cdots, w_n)^T$ should be induced by minimizing $f(w_1, w_2, \cdots, w_n)$. This is rather difficult to solve because the objective function is nonlinear and usually nonconvex, moreover, no unique solution exists and the solutions are not easily computable.

III. NEW METHODS

A. The ideas of new methods

In least square method, the error is $a_{ij} - w_i / w_j$. The expression $a_{ij} - w_i / w_j$ is nonlinear, thus the least square problem is nonlinear programming. If the error is $a_{ij} w_j - w_i$, the expression is linear. We can not only use the sum of squares of error as objective function, but also use the sum of error absolute value and the error absolute value as objective function. Three methods are given as follows.

B. New Least Squares method

Using sum of squares of error as objective function, the model is

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} w_j - w_i)^2$$

$$s.t. \quad \sum_{i=1}^{n} w_i = 1 \qquad (4)$$

$$w_i \ge 0, i = 1,2,\cdots,n$$

Thus, we can construct Lagrange function

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} w_j - w_i)^2 + \lambda(\sum_{i=1}^{n} w_i - 1)$$

Where is the Lagrange multiplier.

$$\partial L / \partial w_i = -2(a_{i1} w_1 - w_i) - 2(a_{i2} w_2 - w_i) - \cdots - 2(a_n w_n - w_i) + 2a_{1i}(a_{1i} w_i - w_1) + 2a_{2i}(a_{2i} w_i - w_2) + \cdots + 2a_{ni}(a_{ni} w_i - w_n) + \lambda$$

$$= -2(a_{i1} + a_{1i})w_1 - 2(a_{i2} + a_{2i})w_2 - \cdots + [2(n-1) + 2\sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ji}^2]w_i - \cdots - 2(a_{in} + a_{ni})w_n + \lambda$$

Let $\partial L / \partial w_i = 0$ ($i = 1,2,\cdots,n$), the result are

$$-2(a_{i1} + a_{1i})w_1 - 2(a_{i2} + a_{2i})w_2 - \cdots + [2(n-1) + 2\sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ji}^2]w_i - \cdots - 2(a_{in} + a_{ni})w_n + \lambda = 0$$

$$(i = 1,2,\cdots,n)$$

Add $\sum_{i=1}^{n} w_i = 1$, we have linear system about $n+1$ equations. Solve the linear system, we obtain $w_1, w_2, \cdots, w_n$ and λ.

C. Minimax method

Using maximum error absolute value as objective function, the model is

$$\min \max_{1 \le i, j \le 1} |a_{ij} w_j - w_i|$$

$$s.t. \quad \sum_{i=1}^{n} w_i = 1 \qquad (5)$$

$$w_i \ge 0, i = 1,2,\cdots,n$$

Let $v = \max_{1 \le i, j \le 1} |a_{ij} w_j - w_i|$, model (5) is translated into

$$\min v$$
$$s.t.\quad \sum_{i=1}^{n} w_i = 1$$
$$a_{ij}w_j - w_i \le v(i = 1,2,\cdots,n;\ j = 1,2,\cdots,n) \tag{6}$$
$$a_{ij}w_j - w_i \ge -v(i = 1,2,\cdots,n;\ j = 1,2,\cdots,n)$$
$$v \ge 0, w_i \ge 0, i = 1,2,\cdots,n$$

This is a linear programming. $w_i(i = 0,1,\cdots,l)$ can be get by simplex method [13].

D. Absolute Deviation Method

Using the sum of error absolute value as objective function, the model is

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} \left|a_{ij}w_j - w_i\right|$$
$$s.t.\quad \sum_{i=1}^{n} w_i = 1 \tag{7}$$
$$w_i \ge 0, i = 1,2,\cdots,n$$

Let $u_{ij} = \begin{cases} a_{ij}w_j - w_i & a_{ij}w_j > w_i \\ 0 & a_{ij}w_j \le w_i \end{cases}$, $v_{ij} = \begin{cases} 0 & a_{ij}w_j > w_i \\ -a_{ij}w_j + w_i & a_{ij}w_j \le w_i \end{cases}$, （$i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$）

So $u_{ij} - v_{ij} = a_{ij}w_j - w_i$, $\left|a_{ij}w_j - w_i\right| = u_{ij} + v_{ij}$, $u_{ij} \ge 0, v_{ij} \ge 0$ （$i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ ）,and $u_{ii} = v_{ii} = 0$ （$i = 1,2,\cdots,n$）.

The model (7) is translated into

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}(u_{ij} + v_{ij})$$
$$s.t.\quad \sum_{i=1}^{n} w_i = 1$$
$$u_{ij} - v_{ij} = a_{ij}w_j - w_i, i = 1,2,\cdots,n;\ j = 1,2,\cdots,n \tag{8}$$
$$w_i \ge 0, i = 1,2,\cdots,n$$
$$u_{ij} \ge 0, v_{ij} \ge 0, i = 1,2,\cdots,n;\ j = 1,2,\cdots,n$$

This is a linear programming also.

IV. Incomplete AHP

However, in some real problems, it is impossible or difficult to have comparisons of some pairs of alternatives. Let us call such cases incomplete AHP. It is very important to estimate incomplete comparisons data to have alternative's weights. The typical methods in incomplete AHP are Two-Stage method [14-15] and Harker method[16]. In Harker method, however, weights are calculated without estimate unknown comparisons. In Two-Stage method, estimation for unknown comparisons is carried out, but the priority of known comparisons and estimated comparisons are treated with equal importance. Two-Stage method presents a method for estimating a missing datum of an incomplete matrix.

A. Harker's Method

Harker method is based on the following idea. If (i, j) -component is missing, put the artificial value w_i/w_j into the vacant component to construct a complete reciprocal matrix $A(w)$. Then consider the eigensystem problem:

$$A(w)w = \lambda w.$$

Formally, Harker's method is written as follows. Given incomplete matrix $A = (a_{ij})$, define the corresponding *derived reciprocal matrix* $\tilde{A} = (a_{ij})$ by

$$\tilde{a}_{ij} = \begin{cases} 1 + m_i & \text{if} \quad i = j \\ 0 & \text{if} \quad a_{ij} \quad \text{is missing} \\ a_{ij} & \text{otherwise} \end{cases}$$

where m_i denotes the number of missing components in the i -th row.

The Harker's algorithms can be described as follows:

Step 1 Construct a derived reciprocal matrix $\tilde{A}$ of $A(x)$.

Step 2 Calculate the largest eigenvalue $\tilde{\lambda}_{\max}$ of $\tilde{A}$ and its associate eigenvector.

Step 3 Normalize the eigenvector into a priority weight vector.

B. Logarithmic Least Squares method

Using sum of logarithmic squares of error as objective function, the model is

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij}(\ln a_{ij} + \ln w_j - \ln w_i)^2$$
$$s.t.\quad \sum_{i=1}^{n} w_i = 1 \tag{9}$$
$$w_i \ge 0, i = 1,2,\cdots,n$$

where, $\delta_{ij} = \begin{cases} 0 & a_{ij}\ is \quad \text{missing} \\ 1 & \text{otherwise} \end{cases}$.

Let $r_{ij} = \ln a_{ij}, x_i = \ln w_i - \beta$, the model（9） is translated into

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}(r_{ij}+x_j-x_i)^2$$
$$s.t.\quad \sum_{i=1}^{n} e^{x_i+\beta}=1$$

By solving the above minimization problem, the weight vector W is described as follows vertical equation:

$$x_i\sum_{j=1}^{n}\delta_{ij}-\sum_{j=1}^{n}\delta_{ij}x_j=\sum_{j=1}^{n}\delta_{ij}r_{ij}\,(i=1,2,\cdots,n)$$

We normalize the above weight vector, the weight vector is:

$$w_i=\frac{e^{x_i}}{\sum_{j=1}^{n}e^{x_j}}$$

C. New Least Squares Method for Incomplete Matrices

We can apply proposed methods to the ranking estimation Incomplete AHP.

Using sum of squares of error as objective function, the model is

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}(a_{ij}w_j-w_i)^2$$
$$s.t.\quad \sum_{i=1}^{n} w_i=1 \qquad (10)$$
$$w_i\ge 0,\, i=1,2,\cdots,n$$

where, $\delta_{ij}=\begin{cases}0 & a_{ij}\ is\ \text{missing}\\ 1 & \text{otherwise}\end{cases}$

Thus, we can construct Lagrange function

$$L=\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}(a_{ij}w_j-w_i)^2+\lambda(\sum_{i=1}^{n}w_i-1)$$

Where is the Lagrange multiplier.

$$\partial L/\partial w_i=-2\delta_{i1}(a_{i1}w_1-w_i)-2\delta_{i2}(a_{i2}w_2-w_i)-$$
$$\cdots-2\delta_{in}(a_nw_n-w_i)+2\delta_{1i}a_{1i}(a_{1i}w_i-w_1)+$$
$$2\delta_{2i}a_{2i}(a_{2i}w_i-w_2)+\cdots+2\delta_{ni}a_{ni}(a_{ni}w_i-w_n)+\lambda$$
$$=-2(\delta_{i1}a_{i1}+\delta_{1i}a_{1i})w_1-2(\delta_{i2}a_{i2}+\delta_{2i}a_{2i})w_2-\cdots$$
$$+2[\sum_{\substack{j=1\\ j\ne i}}^{n}\delta_{ij}+\sum_{\substack{j=1\\ j\ne i}}^{n}(\delta_{ji}a_{ji}^2)]w_i-2(\delta_{i,i+1}a_{i,i+1}+\delta_{i,i+1}a_{i,i+1})w_{i+1}$$
$$\cdots-2(\delta_{in}a_{in}+\delta_{n1}a_{ni})w_n+\lambda$$

Let $\partial L/\partial w_i=0$ （$i=1,2,\cdots,n$）, the result are

$$-2(\delta_{i1}a_{i1}+\delta_{1i}a_{1i})w_1-2(\delta_{i2}a_{i2}+\delta_{2i}a_{2i})w_2-\cdots$$
$$+2[\sum_{\substack{j=1\\ j\ne i}}^{n}\delta_{ij}+\sum_{\substack{j=1\\ j\ne i}}^{n}(\delta_{ji}a_{ji}^2)]w_i-\cdots-2(\delta_{in}a_{in}+\delta_{n1}a_{ni})w_n$$
$$+\lambda=0$$
$$(i=1,2,\cdots,n)$$

Add $\sum_{i=1}^{n}w_i=1$, we have linear system about $n+1$ equations. Solve the linear system, we obtain $w_1,w_2,\cdots,w_n$ and λ.

D. MinimaxMethod for Incomplete Matrices

Using maximum error absolute value as objective function, the model is

$$\min \max_{1\le i,j\le n}\delta_{ij}\left|a_{ij}w_j-w_i\right|$$
$$s.t.\quad \sum_{i=1}^{n}w_i=1 \qquad (11)$$
$$w_i\ge 0,\, i=1,2,\cdots,n$$

Let $v=\max_{1\le i,j\le n}\delta_{ij}\left|a_{ij}w_j-w_i\right|$, the model（11） is translated into

$$\min v$$
$$s.t.\quad \sum_{i=1}^{n}w_i=1$$
$$\delta_{ij}\left|a_{ij}w_j-w_i\right|\le v(i=1,2,\cdots,n; \qquad (12)$$
$$j=1,2,\cdots,n)$$
$$v\ge 0, w_i\ge 0, i=1,2,\cdots,n$$

This is a linear programming. $w_i(i=0,1,\cdots,l)$ can be get by simplex method.

E. Absolute Deviation Method for Incomplete Matrices

Using the sum of error absolute value as objective function, the model is

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}\left|a_{ij}w_j-w_i\right| \qquad (13)$$
$$s.t.\quad \sum_{i=1}^{n}w_i=1 \qquad w_i\ge 0, i=1,2,\cdots,n$$

Let $u_{ij}=\begin{cases}a_{ij}w_j-w_i & a_{ij}w_j>w_i\\ 0 & a_{ij}w_j\le w_i\end{cases}$,

$v_{ij}=\begin{cases}0 & a_{ij}w_j>w_i\\ -a_{ij}w_j+w_i & a_{ij}w_j\le w_i\end{cases}$, （$i=1,2,\cdots,n$, $j=1,2,\cdots,n$）

So $u_{ij} - v_{ij} = a_{ij} w_j - w_i$, $\left| a_{ij} w_j - w_i \right| = u_{ij} + v_{ij}$, $u_{ij} \ge 0, v_{ij} \ge 0$ ($i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$), and $u_{ii} = v_{ii} = 0$ ($i = 1,2,\cdots,n$).

The model (13) is translated into

$$
\begin{aligned}
&\min \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} (u_{ij} + v_{ij}) \\
&s.t. \quad \sum_{i=1}^{n} w_i = 1 \\
&\qquad u_{ij} - v_{ij} = a_{ij} w_j - w_i , i = 1,2,\cdots,n; \\
&\qquad\qquad j = 1,2,\cdots,n \\
&\qquad w_i \ge 0 , i = 1,2,\cdots,n \\
&\qquad u_{ij} \ge 0, v_{ij} \ge 0, i = 1,2,\cdots,n; j = 1,2,\cdots,n
\end{aligned}
\tag{14}
$$

This is a linear programming also.

V. Numerical Examples

A.. *Complete Matrice*

Suppose that following is the judgement matrix[12]:

$$
A = \begin{bmatrix} 1 & \frac{1}{2} & 4 & 3 & 3 \\ 2 & 1 & 7 & 5 & 5 \\ \frac{1}{4} & \frac{1}{7} & 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{5} & 2 & 1 & 1 \\ \frac{1}{3} & \frac{1}{5} & 3 & 1 & 1 \end{bmatrix}
$$

a. Using new Least Squares method, we have equation as follows:

$$
\begin{bmatrix} 16.5694 & -5 & -8.5 & -6.6667 & -6.6667 & 1 \\ -5 & 8.7008 & -14.2857 & -10.4 & -10.4 & 1 \\ -8.5 & -14.2857 & 164 & -5 & -6.6667 & 1 \\ -6.6667 & -10.4 & -5 & 78.5 & -4 & 1 \\ -6.6667 & -10.4 & -6.6667 & -4 & 78.2222 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

The tables 1 presents the of the simulation's output.

b. Using Minimax method, we have linear programming as follows:

$$
\begin{aligned}
&\min v \\
&s.t. \quad w_1 + w_2 + \cdots + w_5 = 1 \\
&-w_1 + \tfrac{1}{2} w_2 - v \le 0 \\
&w_1 - \tfrac{1}{2} w_2 - v \le 0 \\
&-w_1 + 4w_3 - v \le 0 \\
&w_1 - 4w_3 - v \le 0 \\
&-w_1 + 3w_4 - v \le 0 \\
&w_1 - 3w_4 - v \le 0 \\
&-w_1 + 3w_5 - v \le 0 \\
&w_1 - 3w_5 - v \le 0 \\
&2w_1 - w_2 - v \le 0 \\
&-2w_1 + w_2 - v \le 0 \\
&-w_2 + 7w_3 - v \le 0 \\
&w_2 - 7w_3 - v \le 0 \\
&-w_2 + 5w_4 - v \le 0 \\
&w_2 - 5w_4 - v \le 0 \\
&-w_2 + 5w_5 - v \le 0 \\
&w_2 - 5w_3 - v \le 0 \\
&\tfrac{1}{4} w_1 - w_3 - v \le 0 \\
&-\tfrac{1}{4} w_1 + w_3 - v \le 0 \\
&\tfrac{1}{7} w_2 - w_3 - v \le 0 \\
&-\tfrac{1}{7} w_2 + w_3 - v \le 0 \\
&-w_3 + \tfrac{1}{2} w_4 - v \le 0 \\
&w_3 - \tfrac{1}{2} w_4 - v \le 0 \\
&-w_3 + \tfrac{1}{3} w_5 - v \le 0 \\
&w_3 - \tfrac{1}{3} w_5 - v \le 0 \\
&\tfrac{1}{3} w_1 - w_4 - v \le 0 \\
&-\tfrac{1}{3} w_1 + w_4 - v \le 0 \\
&\tfrac{1}{5} w_2 - w_4 - v \le 0 \\
&-\tfrac{1}{5} w_2 + w_4 - v \le 0 \\
&2w_3 - w_4 - v \le 0 \\
&-2w_3 + w_4 - v \le 0 \\
&-w_4 + w_5 - v \le 0 \\
&w_4 - w_5 - v \le 0 \\
&\tfrac{1}{3} w_1 - w_5 - v \le 0 \\
&-\tfrac{1}{3} w_1 + w_5 - v \le 0 \\
&\tfrac{1}{5} w_2 - w_5 - v \le 0 \\
&-\tfrac{1}{5} w_2 + w_5 - v \le 0 \\
&3w_3 - w_5 - v \le 0 \\
&-3w_3 + w_5 - v \le 0 \\
&w_4 - w_5 - v \le 0 \\
&-w_4 + w_5 - v \le 0 \\
&v \ge 0, w_i \ge 0, i = 1,2,\cdots,n .
\end{aligned}
$$

To solve this linear programming, a software optimization of Matlab is utilized. Table 1 illustrate the comparison of methods.

c. Using absolute deviation method, we have linear programming as follows:

$$
\min u_{12} + v_{12} + u_{13} + v_{13} + \cdots + u_{54} + v_{54}
$$

$$
\begin{aligned}
s.t. \quad & w_1 + w_2 + \cdots + w_5 = 1 \\
& u_{12} - v_{12} + w_1 - \tfrac{1}{2} w_2 = 0 \\
& u_{13} - v_{13} + w_1 - 4w_3 = 0 \\
& u_{14} - v_{14} + w_1 - 3w_4 = 0 \\
& u_{15} - v_{15} + w_1 - 3w_5 = 0 \\
& u_{21} - v_{21} + w_2 - 2w_1 = 0 \\
& u_{23} - v_{23} + w_2 - 7w_3 = 0 \\
& u_{24} - v_{24} + w_2 - 5w_4 = 0 \\
& u_{25} - v_{25} + w_2 - 5w_5 = 0 \\
& u_{31} - v_{31} + w_3 - \tfrac{1}{4} w_1 = 0 \\
& u_{32} - v_{32} + w_3 - \tfrac{1}{7} w_2 = 0 \\
& u_{34} - v_{34} + w_3 - \tfrac{1}{2} w_4 = 0 \\
& u_{35} - v_{35} + w_3 - \tfrac{1}{3} w_5 = 0 \\
& u_{41} - v_{41} + w_4 - \tfrac{1}{3} w_1 = 0 \\
& u_{42} - v_{42} + w_4 - \tfrac{1}{5} w_2 = 0 \\
& u_{43} - v_{43} + w_4 - 2w_3 = 0 \\
& u_{45} - v_{45} + w_4 - w_5 = 0 \\
& u_{51} - v_{51} + w_5 - \tfrac{1}{3} w_1 = 0 \\
& u_{52} - v_{52} + w_5 - \tfrac{1}{5} w_2 = 0 \\
& u_{53} - v_{53} + w_5 - 3w_3 = 0 \\
& u_{54} - v_{54} + w_5 - w_4 = 0 \\
& w_i \ge 0, i = 1,2,\cdots,n \\
& u_{ij} \ge 0, v_{ij} \ge 0, i = 1,2,\cdots,n;\ j = 1,2,\cdots,n .
\end{aligned}
$$

Table 1 illustrate the comparison of methods.

TABLE I. COMPARISON OF SIX SOLUTION METHODS TO THE ANALYTIC HIERARCHY PROCESS

Methods	（$w_1, w_2, \cdots, w_5$）
Sum method	（0.2623,0.4744,0.0545,0.0985,0.1103）
Geometric mean method	（0.2636,0.4773,0.0531,0.0988,0.1072）
Eigenvector method	（0.2636,0.4758,0.0538,0.0981,0.1087）
New least squares method	（0.2584,0.4859,0.0628,0.0957,0.0973）
Minimax method	（0.2653,0.4653,0.0571,0.1061,0.1061）
Absolute deviation method	（0.2703,0.4730,0.0676,0.0946,0.0946）

We have demonstrated the use of new least squares method, Minimax method and absolute deviation method for deriving the priority weights assessment in the AHP method as another alternative for the originally technique of eigenvector method of Saaty. The six solution approaches to the AHP problem are nearer. Sum method and geometric mean method are easy to handle in calculation, but new least squares method, Minimax method and absolute deviation method with results more reasonable are more reasonable.

B. *Incomplete Matrice*

Suppose that following is the incomplete matrix：

$$
A = \begin{bmatrix} 1 & 3 & 6 & * & \frac{1}{4} \\ \frac{1}{3} & 1 & 2 & 1 & * \\ \frac{1}{6} & \frac{1}{2} & 1 & \frac{1}{2} & * \\ * & 1 & 2 & 1 & 2 \\ 4 & * & * & \frac{1}{2} & 1 \end{bmatrix}
$$

Where * is a missing entry.

a. Harker's Method

The derived reciprocal matrix $\widetilde{A}$ is

$$
\widetilde{A} = \begin{bmatrix} 2 & 3 & 6 & 0 & \frac{1}{4} \\ \frac{1}{3} & 2 & 2 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 4 & 0 & 0 & \frac{1}{2} & 3 \end{bmatrix}
$$

From $\widetilde{A}$, we calculate the principal eigenvalue $\lambda_{\max} = 5.7459$, and principal eigen vector w =(0.2130, 0.1181, 0.0591, 0.2534, 0.3564).

b. Logarithmic Least Squares method

The vertical equation is described as follows:

$$
\begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \ln \frac{9}{2} \\ \ln \frac{2}{3} \\ -\ln 24 \\ 2\ln 2 \\ \ln 2 \end{bmatrix}
$$

The vector weight is w =(0.2342, 0.1627, 0.0899, 0.2362, 0.2823).

The tables 2 presents the of the simulation’s output.

c. Using new Least Squares method, we have equation as follows:

$$
\begin{bmatrix} 32.2778 & -6.6667 & -12.3363 & 0 & -8.5 & 1 \\ -6.6667 & 26.5 & -5 & -4 & 0 & 1 \\ -12.3333 & -5 & 94 & -5 & 0 & 1 \\ 0 & -4 & -5 & 9 & -5 & 1 \\ -8.5 & 0 & 0 & -5 & 12.125 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

The tables 2 presents the of the simulation’s output.

d. Using Minimax method**,** we have linear programming as follows:

$$
\begin{aligned}
& \min v \\
s.t. \quad & w_1 + w_2 + \cdots + w_5 = 1 \\
& -w_1 + 3w_2 - v \le 0 \\
& w_1 - 3w_2 - v \le 0 \\
& -w_1 + 6w_3 - v \le 0 \\
& w_1 - 6w_3 - v \le 0
\end{aligned}
$$

$$-w_1+\tfrac{1}{4}w_5-v\le 0$$
$$w_1-\tfrac{1}{4}w_5-v\le 0$$
$$\tfrac{1}{3}w_1-w_2-v\le 0$$
$$-\tfrac{1}{3}w_1+w_2-v\le 0$$
$$-w_2+2w_3-v\le 0$$
$$w_2-2w_3-v\le 0$$
$$-w_2+w_4-v\le 0$$
$$w_2-w_4-v\le 0$$
$$\tfrac{1}{6}w_1-w_3-v\le 0$$
$$-\tfrac{1}{6}w_1+w_3-v\le 0$$
$$\tfrac{1}{2}w_2-w_3-v\le 0$$
$$-\tfrac{1}{2}w_2+w_3-v\le 0$$
$$-w_3+\tfrac{1}{2}w_4-v\le 0$$
$$w_3-\tfrac{1}{2}w_4-v\le 0$$
$$w_2-w_4-v\le 0$$
$$-w_2+w_4-v\le 0$$
$$2w_3-w_4-v\le 0$$
$$-2w_3+w_4-v\le 0$$
$$-w_4+2w_5-v\le 0$$
$$w_4-2w_5-v\le 0$$
$$4w_1-w_5-v\le 0$$
$$-4w_1+w_5-v\le 0$$
$$\tfrac{1}{2}w_4-w_5-v\le 0$$
$$-\tfrac{1}{2}w_4+w_5-v\le 0$$
$$v\ge 0, w_i\ge 0, i=1,2,\cdots,n\,.$$

Table 2 illustrates the comparison of methods.

e. Using absolute deviation method, we have linear programming as follows:

$$\min\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}(u_{ij}+v_{ij})$$
$$s.t.\quad w_1+w_2+\cdots+w_5=1$$
$$u_{12}-v_{12}+w_1-3w_2=0$$
$$u_{13}-v_{13}+w_1-6w_3=0$$
$$u_{15}-v_{15}+w_1-\tfrac{1}{4}w_5=0$$
$$u_{21}-v_{21}+w_2-\tfrac{1}{3}w_1=0$$
$$u_{23}-v_{23}+w_2-2w_3=0$$
$$u_{24}-v_{24}+w_2-w_4=0$$
$$u_{31}-v_{31}+w_3-\tfrac{1}{6}w_1=0$$
$$u_{32}-v_{32}+w_3-\tfrac{1}{2}w_2=0$$
$$u_{34}-v_{34}+w_3-\tfrac{1}{2}w_4=0$$
$$u_{42}-v_{42}+w_4-w_2=0$$
$$u_{43}-v_{43}+w_4-2w_3=0$$
$$u_{45}-v_{45}+w_4-2w_5=0$$
$$u_{51}-v_{51}+w_5-4w_1=0$$
$$u_{54}-v_{54}+w_5-\tfrac{1}{2}w_4=0$$
$$w_i\ge 0, i=1,2,\cdots,n$$
$$u_{ij}\ge 0, v_{ij}\ge 0, i=1,2,\cdots,n;\quad j=1,2,\cdots,n\,.$$

Table 2 illustrates the comparison of methods.

TABLE II. COMPARISON OF THREE SOLUTION METHODS TO INCOMPLETE MATRICES

Methods	$(w_1, w_2, \cdots, w_5)$
Harker's Method	(0.2130,0.1181,0.0591,0.2534,0.3564)
Logarithmic Least Squares method	(0.2342,0.1627,0.0899,0.2362,0.2823)
New least squares method	(0.1339,0.1322,0.0534,0.3634,0.3171)
Minimax method	(0.1368,0.1263,0.0632,0.3684,0.3053)
Absolute deviation method	(0.1714,0.057,0.0286,0.0571,0.6857)

VI. CONCLUSIONS

The traditional Least square method is a nonlinear programming. New least squares method is translated into linear system and Minimax method and absolute deviation method are translated into linear programming. It is shown that three methods proposed in this paper have fast convergence and smaller computational complexity. New proposed methods can also apply to the ranking estimation in incomplete AHP. It is very important to estimate incomplete comparisons data to have alternative's weights.

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REFERENCES

[1] T.L. Saaty, "A Scaling Method for Priorities in Hierarchical Structures", Journal of Mathematical Psychology, 1977, vol.15, no.3, pp.234-281.

[2] T.L. Saaty, Fundamentals of Decision Making and Priority Theory with the AHP. RWS Publications, Pittsburgh, PA, USA, 1994.

[3] P. Putrus, "Accounting for Intangibles in Integrated Manufacturing", Financial and Accounting Systems, 1991, vol.7, no.1, pp.30-35.

[4] T.O. Boucher and E.L. McStravic, "Multi-attribute Evaluation Within a Present Value Framework and its Relation to the Analytic Hierarchy Process", The Engineering Economist, 1991,vol.37. no.1, pp.1-32.

[5] R.N. Wabalickis, "Justification of FMS With the Analytic Hierarchy Process", Journal of Manufacturing Systems, 1988,vol.17, no.3, pp.175-182.

[6] K.E. Cambron and G.W. Evans, “Layout Design Using the Analytic Hierarchy Process”, Computers & IE, 1991,vol.20, no.2,pp. 221-229.

[7] L. Wang, and T. Raz, “Analytic Hierarchy Process Based on Data Flow Problem”, Computers & IE, 1991, vol.20, no.3,pp.355-365.

[8] B.Q. Chen, “Two New Priority Methods in Analytic Hierarchy Process”, Journal of Systems Engineering, 1990,vol.5,no.2, pp.43-51(in Chinese).

[9] Y.M. Wang and G.W. Fu, “Class of generalized least deviations priority methods of comparison matrix in analytic hierarchy process”, Journal of Tsinghua University. 1993,vol.33, no.3,pp.10-17(in Chinese).

[10] Z.M. Zhang and D.M. Li,. “The Properties of Least Deviations Method”, Systems Engineering -Theory & Practice, 1996,vol.16, no.7, pp.36－39. (in Chinese).

[11] Z.M. Zhang and X.G. Cheng, “The Mlsm of Comparison Matrix”, Journal of Qufu Normal University. 1997,vol. 23,no.1,pp. 17-20(in Chinese).

[12] H.X. Li, “Dominance Matrix Method for Calculating Priority Vector in AHP”, Operations Research and Management Science, 2003,vol.12,no.1, pp.22 － 27 (in Chinese).

[13] R. L. Burden and J. D. Faires. Numerical Analysis. Higher Education Press & Thomson Learning, Inc., 2001.

[14] I. akahashi, “HP Applied to Binary and Ternary Comparisons” Journal of Operations Research Society of Japan, Vol. 33, No. 3, (1990) 199-206.

[15] I.Takahashi and M. Fukuda, “mparisons of AHP with other methods in binary paired comparisons” roceedings of the Second Conference of the Association of Asian-Pacific Operational Research Societies within IFORS, (1991) 325-331.

[16] Harker P. T, “complete Pairwise Comparisons in the Analytic Hierarchy Process” Math. modeling,Vol.9, (1987) 838-848.

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