Intersection between Bi-cubic Bezier Curved Surface and Plane

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Abstract—The paper presents a solution to calculate intersection between a plane and bi-cubic Bezier curved surface. The paper first constructs bounding box based on the control grid by the characteristics of the Bezier surface, and then makes use of the plane half-space properties to convert the problem that calculates the intersection line between the plane and the parameter curved surface into calculating a series of intersection points by using the method that apply the bi-linear interpolation algorithm to the curved surface in representation of parameter area. The intersection line between the plane and the surface can be calculated by connecting the intersection points. This method can solve the surface intersection line well whether the line is discontinuous or form a closed cycle. The algorithm is relatively simple, and has good adaptability for calculating the intersection line between the plane and the curved surface.

Index Terms—bounding box, Linear interpolation, Bezier surface

I.  INTRODUCTION

Along with the free surface in geometric modeling, CAD/CAM, RE applications, Bezier surface in various fields have occupied a very important position, and then on the surface of a variety of problems that follow close on succession. In particular, Bezier surface and plane intersection became a popular research topic in computer geometry. Usually algorithm which is high efficiency, computational stability, and accuracy easy to control is required. The metallurgical phase diagram is very important to our study surfaces, bi-cubic Bezier surfaces, the surfaces by three spatial variables and two change parameters within the range [0, 1] is based on a five-dimensional space representation[1-2]. But simply calculated using mathematical methods are difficult or even impossible, and also exists the problem of the accuracy and efficiency with a non-mathematical methods or non-pure mathematical methods. Therefore, how accurate, stable, fast solution to the intersection of parametric surfaces has been related topics at the forefront of the field of study.

The existing literatures on the intersection calculation between the parametric surface and plane had a lot of researches, and proposed several intersection algorithms, as the following methods:

(1) According to the parameter domain bidirectional interpolation principle, calculating the near-surface point tracking method [3-7]. This method is first to calculate proximal point of the surface as the initial point of intersection, and then starting from the initial known point. And next point can be calculated in the same way, and then a method for calculating the whole intersection will be solved.

This method is relatively simple to calculate the intersection points between the boundary lines with the plane. But in the calculation of curved surface and the plane of intersection is more complex and the tracking direction and trace step are difficult to determine, Especially how to calculate the tracking direction under the condition of singular case is a difficult problem, which increase the uncertainty about intersection and the accuracy is greatly reduced;

(2) Discrete segmentation method [8-11]. The method makes the surface discrete into smaller patches, until each patch can be relatively simple surfaces (such as quadrilateral or triangular planar). And then use these simple patches to obtain a series of line segment intersection, connect these segments to get approximate accurate intersection The method is relatively easy to be programed and of high reliability, But it has low accuracy, and a large amount of storage; If high precision is requirements, it needs to increase the discrete layers, which greatly increased the data storage and computation;

(3) The intersection calculation of the plane vector field, determine the initial point of intersection with the plane vector field and tracked from the intersection point [12]. The method has a unique advantage in resolving cross-line cycle, but how to define a suitable plane vector field is the key to solve the problem;

(4) "Sub" method base on control meshes [13-14]. The method uses a small facet which controls grid the
approximate instead of the surface itself to calculate, there is a series of problems in the accuracy.

(5) Intersection algorithm based on the control point \[15\]. The method uses the control points of the Bezier surface patches to construct scalable ruled dough surface patches. It develops the problem of Bezier surface intersection into the problem of the scalable ruled surface intersection, the algorithm applies only to the simple triangular Bezier surface. The applicability of the algorithm is bad.

The paper presents an intersection calculation algorithm based on the bounding box. The paper makes full use of the references by analyzing the characteristics of the Bezier surface, and combines the bounding box technology \[16-18\], the nature of the half-space properties, and bi-linear interpolation algorithm. It can simply, quickly find the plane and Bezier surfaces the line of intersection. So the intersection line between the plane and the Bezier curved surface can be calculated simply and quickly.

II. BI-CUBIC BEZIER SURFACE BOUNDARY LINE AND THE PLANE INTERSECTION ALGORITHM

The paper introduced the bounding box technology into the Bezier surface intersection algorithm based on the unique nature of the bi-cubic curved surface. The paper presents an algorithm of calculating intersection between a plane and bi-cubic Bezier curved surface based on the constructed bounding box and the nature of the Bezier curved surface. The algorithm flowchart was given as shown in Fig. 1.

This paper first constructs the bounding box of the curved surface, and it first determines whether exist intersection between the curved surface and the bounding box when determine whether the surface and the plane intersects \[19\]. After determining the bounding box and the plane intersected, then calculate the intersection between the surface boundary line and the plane, determine the entire intersection segment line. When the surface boundary line and the plane without intersection, there exists two conditions, One condition is that the curved surface and the plane without intersection, and the other is that the intersection between the surface and the plane form a closed loop. Next we need to split the curved surface into many small curved surface pieces, then each small curved surface piece needs to be split in the same way, and determine the intersection line between the curved surface and the plane. The following will introduce the specific implementation process.

A. Constructing the Bounding Box

Make full use of the convex hull of Bezier surface \[20\], the surface is formed by sixteen control points to control the shape of the surface to determine, and it is strictly in the control polygon grid. Sixteen control points connected bi-cubic Bezier surfaces to form a control polygon mesh, thereby bounding box formation. As shown in Fig. 2:

The control polygon grid nine surfaces constitute a bounding box. The bounding box in the general sense is different, the bounding boxes are no longer closed front
body structure, but a midair convex polyhedron structure. This paper studies the three Bezier surface of the convex hull, surface located in the control polygon mesh. This method of bounding box structure, structure is relatively simple and intuitive, and the tightness is also very good.

B. The Determination Method of Intersecting

In assessing the intersection plane and curved surface, using the half-space properties. Proposing plane equation \( F(x, y, z) = Ax + By + Cz + D = 0 \), so the Function F divides the space into two parts. The whole part of space satisfied for \( F(x, y, z) > 0 \) may be referred to the upper half plane space. The whole part of space satisfied for \( F(x, y, z) < 0 \) may be referred to the lower half plane space. The whole part of space satisfied for \( F(x, y, z) = 0 \) represents the set of all points located on a plane. Using this property, it is easy to determine the relationship between point, line and the plane.

Specific to the plane and bi-cubic Bezier curved surface intersection, first by judging the plane intersects the bounding box, the position relationship between sixteen points and planes bounding box, so as to judge the relationship of plane and curved surface.

Due to the double three Bezier surface of the convex hull, the intersection of parametric surface boundary line and plane is divided into the following: no intersection scenarios, The only point, The multiple points of intersection situation, The plane cutting surfaces to form a line segment, The plane parallel tangent surface forming a plurality of line segment or a closed loop. The detailed analysis process as shown in Fig. 3:

(1) If the directed distance is the same number, then the bounding box in the side of the plane, and also the plane and bounding box will not intersect, the plane and parametric surfaces do not intersect;

(2) If the directed distance is Zero, then the point is in the plane.

(3) If the directed distance is opposite number, then the bounding box and plane intersect, thereby the problem of determining the plane and curved surfaces intersect may first determine the intersection of the boundary line with the plane, the next given calculation.

(4) If the plane and curved surfaces do not intersect, it needs to further split the parametric surfaces, the details will be given in section 4.

Thereby, it is possible to quickly determine the plane and the possibility of parametric surfaces intersecting, and greatly reduces the workload in the calculation.

C. Intersection of Parametric Surface Boundary Line and Plane

Bi-cubic parameter Bezier surfaces have the properties of endpoint geometry and boundary geometry, and the four corner point controlled polygon grid is just the four corner points of Bezier surface. The points those control polygon grid defines four boundary lines of Bezier surface, as shown in Fig. 2, the boundary line \( P_{00} P_{30} \) is controlled by the control point \( P_{00}, P_{10}, P_{20}, P_{30} \). Specific curve as shown in Fig. 4:

![Figure 4. Cubic Bezier line.](image)

Whether plane and bounding box intersect or not is judged by the direct distance between the control point of the bounding box and plane.

(1) If the directed distance between sixteen control points and the plane is the same number, then the bounding box in the side of the plane, and also the plane and bounding box will not intersect, the plane and parametric surfaces do not intersect;

(2) If the directed distance is Zero, then the point is in the plane.

(3) If the directed distance is opposite number, then the bounding box and plane intersect, thereby the problem of determining the plane and curved surfaces intersect may first determine the intersection of the boundary line with the plane, the next given calculation.

(4) If the plane and curved surfaces do not intersect, it needs to further split the parametric surfaces, the details will be given in section 4.

Thereby, it is possible to quickly determine the plane and the possibility of parametric surfaces intersecting, and greatly reduces the workload in the calculation.
range, and the point \( p_{a1} \) is the intersection of the boundary line and the plane. Otherwise determine the value of \( f(P_{a0}) \times f(P_{a2}) \) and \( f(P_{a1}) \times f(P_{a3}) \) in the same way until calculate the intersection of the boundary line and the plane.

2) If the directed distance between both points \( P_{00}, P_{30} \) and the plane are zero, then both points are in the plane. And next determine the directed distance between both points \( P_{20}, P_{20} \) and the plane, if both are zero, the boundary line is in the plane. If one of the directed distance is zero, now we assume \( f(P_{a0}) = 0 \), and next determine the value of both \( f(P_{a0}) \times f(P_{a2}) \) and \( f(P_{a1}) \times f(P_{a3}) \) in the same way by step 1, and calculate other intersections.

3) If \( f(P_{a0}) \times f(P_{a2}) > 0 \), calculate the distance between the midpoint \( p_{a1} \) and the plane whether is in the accuracy of the control range. If \( p_{a1} \) is the intersection and both \( f(P_{00}) \times f(P_{30}) < 0 \) and \( f(P_{10}) \times f(P_{20}) < 0 \), then two intersections will be calculated by Bipartite midpoint method.

Intersection conditions of the boundary line with the plane can be easily obtained in this way. Doing the same processing to the four boundary line, the intersection conditions of the plane and the four boundary lines can be determined. This way is a convenient and quick method to calculate the intersection between the boundary lines and the plane.

D. Internal Intersection Calculation of Surface

For the plane through the parametric surface formed a line segment, as shown in Fig. 5:

![Figure 5. The plane through the parametric surface to form a line.](image)

For the curved surface internal intersection, we use the linear interpolation method to calculate. For example, for the boundary line \( P_{00}, P_{30} \), we use a parameter \( v \) which ranges from 0 to 1, and then choose a fixed number \( t \), calculating two points \( P_{10}, P_{20} \) in the boundary line \( P_{00}, P_{30} \) by Bezier surface matrix expression. The expression as shown in (1).

\[
P(u, v) = [B_{0,0}(u), B_{1,0}(u), \ldots, B_{n,0}(u)]
\]

We can calculate the intersection \( p_{11} \) by in Bipartite midpoint method for the two points \( p_{01}, p_{11} \). And then we increase \( v \) by a certain step size \( \Delta v = t \), when it is a sufficiently small value, we may be considered that the line of intersection is a straight line segment within the range of \( \Delta v \), as shown in Fig. 6:

![Figure 6. Curved surface intersection calculation.](image)

We increase \( u \) and \( v \) by the value \( \Delta u \) and \( \Delta v \), and choose \( \Delta d \) in a certain range, for the chosen value \( t \) is sufficiently small, so in the intersection approximation of a small straight line segment, the next intersection must be in the range of \( \Delta d \). In the condition of \( 0 < u < 1 \) and \( 0 < v < 1 \), when \( v = v + t \) we calculate the intersection between the plane and the curved surface in the same way, and we will calculate a series of intersections with the cycle calculation. Because of the sufficiently small value \( t \), we use the obtained intersections to connect into a line, and we can think of the line as the intersection between the plane and the curved surface.

III. THE CLOSED OR MULTI ARTICLE INTERMITTENT INTERSECTION CALCULATION

When calculating intersection between the plane and Bezier surface, it sometime forms a closed or multi article intermittent intersection, as shown in Fig. 7:

![Figure 7. Intersection type.](image)

The paper chooses the segmentation algorithm to calculate intersection [21-22], specifically segmentation shown in Fig. 8:

\[
\begin{bmatrix}
P_{00} & P_{01} & \cdots & P_{0t} \\
P_{10} & P_{11} & \cdots & P_{1t} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n0} & P_{n1} & \cdots & P_{nt}
\end{bmatrix}
\begin{bmatrix}
B_{0,0}(v) \\
B_{1,0}(v) \\
\vdots \\
B_{n,0}(v)
\end{bmatrix}
\]

(1)
In the condition of given parameter pair \((u, v)\), we choose parameters \(\frac{2}{1} = vu\), and use de-Casteljau algorithm, we can get the equation, as shown in (2).

\[
P(u, v) = \cdots = \sum_{k=0}^{n} \sum_{j=0}^{n-1} P_{i,j,k} B_{i,n}(u)B_{j,n}(v) = \cdots = P_{00}^{n-n}, \quad u, v \in [0,1]
\]

\(P_{i,j,k}^{k,l}\) in (2) can be calculated as (3) or (4) shown.

\[
P_{i,j,k}^{k,l} = \begin{cases} 
(1 - u) P_{i,j}^{k-1,0} + u P_{i+1,j}^{k-1,0}, & (k = 1,2, \cdots, m; \quad l = 0) \\
(1 - v) P_{0,j}^{n-1,l} + v P_{0,j+1}^{n-1,l}, & (k = 0; \quad l = 1,2, \cdots, n) 
\end{cases}
\]

or

\[
P_{i,j,k}^{k,l} = \begin{cases} 
(1 - v) P_{i,j}^{0,l-1} + v P_{i,j+1}^{0,l-1}, & (k = 0; \quad l = 1,2, \cdots, n) \\
(1 - u) P_{i,0}^{k-1,n} + u P_{i+1,0}^{k-1,n}, & (k = 1,2, \cdots, m; \quad l = n) 
\end{cases}
\]

The range of \(i\) and \(j\) in (2) and (3) have been given in (1), and two solutions determine points in the curved surface are given at the same time (as shown in Fig. 9). When processed as (2), we first process curved surface de-Casteljau algorithm on \(n+1\) polygon by parameter \(u\) in the direction \(u\), \(m\) times recurrence, we can get the middle polygon formed by \(n+1\) vertices \(P_{00}^{n-n}\) along the direction \(v\). Then we process curved surface de-Casteljau algorithm by the parameter \(v\), \(n\) times recurrence, the \(P_{00}^{n-n}\) will be got, i.e the point \(P(u, v)\) in the curved surface. Process in the same principal as (3) shown: we first process curved surface de-Casteljau algorithm on \(m+1\) polygon by parameter \(v\) in the direction \(v\), \(n\) times recurrence, the \(P_{00}^{n-n}\) will be got.

The both methods are based on the de-Casteljau algorithm of Bezier curve line, and also can be as Bezier surface definition of equivalent to the recursive definition. Also they can be the basic methods to calculate a point on the Bezier surface. The main feature is the algorithm is stable, reliable, and simple, and the calculation speed is faster.

The bi-cubic curved surface of the paper is controlled by sixteen control points, so \(m = n = 3\). We use de-Casteljau algorithm to calculate the point \(P\) when the parameter \(u = v = 1/2\). At the same time, we also calculate the intersection points \(P_1, P_2, P_3, P_4\) in the boundary line with the given fixed value \(u\) and \(v\). The entire parametric surface is divided into four small surface patches when we connect \(P_1 - P\), \(P_2 - P\), \(P_3 - P\), \(P_4 - P\). Four corner points in each of the four small curved pieces were also identified, and each surface patch bi-cubic Bezier surface characteristics.

For one of small surface patches, we calculate the intersection point between the surface patches boundary line and the plane by the method in section 3.3, and then calculate the intersection segment within the small curved piece by the method in section 3.4. We can calculate all intersection segments of four surface patches. If the bi-cubic Bezier surface and the plane intersect as the picture (a) in Fig. 7, then the intersection segment are the results. If the intersection is a closed circle line as shown in Fig. 7(b), and then all intersection segment will be processed in order, the process steps as follows:

1. First, choose an arbitrary intersection segment as the initial segment, and save in a linked list.
(2) Take any of the remaining intersection segments, and determine the start and end point whether coincide with the saved intersection segment of the linked list. If there are coincident, then save it in the linked list.

(3) Determine whether the intersection segment is the last one, when the start point and the end point of the last intersection, it says that the plane and the curved surface form a closed curved line, and the determine process ends, else turn to step 2.

IV. Test

A test of the proposed algorithm will be given to calculate the intersection points between the plane and the Bezier surface in the way of the author proposed. Assume the plane equation is \( F(x, y, z) = -x + 3y - 2x + 6 = 0 \), the bi-cubic curved sixteen control points as the Table 1 shown.

<table>
<thead>
<tr>
<th>control points</th>
<th>control points</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{00} = (200, 20, 0) )</td>
<td>( P_{0r} = (50, -130, 100) )</td>
</tr>
<tr>
<td>( P_{0t} = (150, 0, 100) )</td>
<td>( P_{0i} = (0, -250, 50) )</td>
</tr>
<tr>
<td>( P_{1r} = (150, 100, 100) )</td>
<td>( P_{1i} = (100, 30, 100) )</td>
</tr>
<tr>
<td>( P_{1r} = (50, -40, 100) )</td>
<td>( P_{1i} = (0, -110, 100) )</td>
</tr>
<tr>
<td>( P_{2r} = (140, 280, 90) )</td>
<td>( P_{2i} = (80, 110, 120) )</td>
</tr>
<tr>
<td>( P_{3r} = (30, 30, 130) )</td>
<td>( P_{3i} = (-50, -100, 150) )</td>
</tr>
<tr>
<td>( P_{4r} = (150, 350, 30) )</td>
<td>( P_{4i} = (50, 200, 150) )</td>
</tr>
<tr>
<td>( P_{5r} = (0, 50, 200) )</td>
<td>( P_{5i} = (-70, 0, 100) )</td>
</tr>
</tbody>
</table>

Through the sixteen control points of the bi-cubic Bezier curved surface are given in table 1, thus the curved surface shape and position can be determined. Known the plane equation and the control points of the Bezier surface, the intersection points can be calculated by programming in the intersection algorithm the paper proposed. And next we connect the points, and the intersection line between the plane and the bi-cubic Bezier curved surface will be drawn. The specific Bezier surface picture and the intersection segment are as shown in Fig. 10, Fig. 11, Fig. 12 and Fig. 13.

The green curved surfaces in the figures are bi-cubic Bezier curved surfaces, and the black line is the intersection segment line. The intersection segment lines show different states under different observed angles.
Fig. 10, Fig. 11, Fig. 12 and Fig. 13 describe the intersection lines in different angles. The Fig. 10 was observed along the X axis direction; Fig. 11 was observed along the tangent plane. Fig. 12 was observed along the X-axis which rotates 30 degrees to the right; and Fig. 13 was observed along the X-axis which rotates 30 degrees to the left.

V. CONCLUSION

The proposed algorithm in the paper combines the bounding box algorithm and linear interpolation algorithm. The introduction of the bounding box, greatly reduced the workload, and it is a quick method to judge whether the plane and the curved surface intersects or not. We can calculate a series of intersection points between the plane and the curved surface by using the method that applies the bi-linear interpolation algorithm. The intersection line between the plane and the surface can be calculated by connecting the intersection points. Through this intersection method, we can avoid complex iterative process for solving differential equations, and in the process of tracking intersection direction of the line of intersection of step uncertainty, but also effectively solve the problem of continuous internal line of intersection of the surface patches. The algorithm is overall relatively simple, and easy to implement, faster computing speed, and it is ideal for solving the intersection line between the plane and curved surface.

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