Subspace Approach for Frequency Estimation of Superimposed Exponential signals in Multiplicative and Additive Noise

Zhihui Liu, Lihua Fu, Shizhong Zhang
School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China
Email: zhihui_liu@foxmail.com

***Abstract*—In this paper, we consider the problem of frequency estimation of superimposed exponential signals in the presence of multiplicative and additive noise. We propose a subspace method based iterative procedure for estimation of signal frequency parameters. The proposed method is based principal eigenvalue vectors of a special constructed data matrix and the weighted least squares (WLS) techniques. Simulations studies are performed to ascertain the performance of the proposed method. It is observed that the proposed method works well in terms of the computational efficiency and estimation accuracy.**

***Index Terms*—Superimposed exponential signals, Multiplicative noise, Frequency estimation, Eigenvalue vectors, Weighted least squares**

I. INTRODUCTION

Estimating the parameters of a superimposed exponential signal in noise from the observed data is a classical but active problem, finding applications in a wide range of areas such as speech signal processing [1-3], biomedical signal processing [4], modeling of biological systems [5], radio location of distant objects [6], and seismic waves processing.

A number of methods for estimation of the parameters of a superimposed exponential signal have been proposed in the past. Notable among these include the Gaussian Maximum Likelihood (GML), Fourier Transform (FT), Modified Forward Backward Linear Prediction (MFBLP) method [7], Estimation of Signal Parameters using Rotational Invariance Technique (ESPRIT) [8], Noise Space Decomposition method (NSD) [9], and so on. However, these methods above consider the parameter estimation under the assumption that harmonics are only contaminated by the additive noise or harmonics with constant amplitude.

It is interesting to observe that in many real life applications, the multiplicative noise may occur, or in other words, the received signals may be random amplitude modulation. For example, in Doppler-radar processing, the knowledge of the frequency from a pulse train reflected from a moving object yields the target's velocity, and it is more appropriate to model the harmonic as having random rather than constant amplitude when the target scintillates [10]. Several methods have been suggested to estimate the parameters of superimposed exponential signals in presence of multiplicative and additive noise, such as cyclic statistics method [10], higher order spectra method [11] and three step iterative method [12]. But the subspace method based iterative procedure for the frequency estimation of a superimposed exponential model with both multiplicative and additive noise has not been considered.

Recently, [13-16] introduced a principal singular-value-vector utilization for model analysis (PUMA) method based iterative procedure for parameter estimation of sinusoidal signals in additive noise. It is observed that such a method works satisfactorily for estimation of the signal parameters in terms of computational complexity and accuracy. The greatest advantage of the method lies in that they make full use of the inner relationship between the weighting matrix for iteration and the parameters to raise the accuracy of the estimators iteratively. Inspired by the works [13-16], in this paper, we generalize the PUMA method to the case of the superimposed exponential signals in the presence of multiplicative and additive noise. It is observed from computer simulations that the proposed method provides fast and accurate frequency estimates at small noise deviation.

The rest of this paper is organized as follows. In Section 2, we present the data model and propose the subspace method for estimating frequencies of superimposed exponential signals in multiplicative and additive noise. Simulation results are provided to evaluate the performance of the proposed method in Section 3. Lastly, the conclusions are drawn in Section 4.

II. PROPOSED FREQUENCY ESTIMATION METHOD

We consider the following model of superimposed exponential signals in multiplicative and additive noise:

$$y(t)=\sum_{k=1}^{p} s_k(t)e^{j(\omega_k t+\varphi_k)}+v(t);\ t=1,2,\cdots,N\ , \qquad (1)$$

where multiplicative noise $\{s_k(t)\}$ is a sequence of independent identically distributed (i.i.d.) random

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variables with finite mean $\mu_k \neq 0$ and variance σ_k^2. Additive noise $\{v(t)\}$ is a sequence of i.i.d. random variables with zero-mean and finite variance σ_v^2. The multiplicative noise and additive noise are assumed to be mutually independent. The number of superimposed signal components, p, is assumed to be known. $\varphi_k \in [-\pi, \pi]$ is the unknown phase. ω_k is the unknown frequencies such that $\{\omega_l \neq \omega_m;\ l \neq m\}$ and $\omega_k \in (-\pi, \pi)$. In this paper, our main purpose is to estimate the unknown frequencies ω_k, given a sample of size N, namely $\{y(1),\ y(2), \cdots, y(N)\}$.

Since $\{s_k(t)\}$ is a sequence of i.i.d. random variables with finite mean μ_k and variance σ_k^2, if we note $\varepsilon_k(t) = s_k(t) - \mu_k$, then $\varepsilon_k(t)$ is a sequence of i.i.d. random variables with zero mean and variance σ_k^2, so we have $s_k(t) = \varepsilon_k(t) + \mu_k$. If we note $N = L \times M$, where L and M are integers, then using (1) we obtain a $L \times M$ matrix $\tilde{\mathbf{X}}$ given by

$$\begin{aligned}\tilde{\mathbf{X}} &= \begin{bmatrix} y(1) & y(L+1) & \cdots & y(L(M-1)+1) \\ y(2) & y(L+2) & \cdots & y(L(M-1)+2) \\ \vdots & \vdots & & \vdots \\ y(L) & y(2L) & \cdots & y(LM) \end{bmatrix} \\ &= \begin{bmatrix} x(1) & x(L+1) & \cdots & x(L(M-1)+1) \\ x(2) & x(L+2) & \cdots & x(L(M-1)+2) \\ \vdots & \vdots & & \vdots \\ x(L) & x(2L) & \cdots & x(LM) \end{bmatrix} \\ &+ \begin{bmatrix} q(1) & q(L+1) & \cdots & q(L(M-1)+1) \\ q(2) & q(L+2) & \cdots & q(L(M-1)+2) \\ \vdots & \vdots & & \vdots \\ q(L) & q(2L) & \cdots & q(LM) \end{bmatrix} \\ &= \mathbf{X} + \mathbf{Q}\end{aligned}, \quad (2)$$

where $\tilde{\mathbf{X}}$ is the noise version of $\mathbf{X}$, $q(t) = \sum_{k=1}^{p} \varepsilon_k(t) e^{j(\omega_k t + \varphi_k)} + v(t)$, $x(t) = \sum_{k=1}^{p} \mu_k e^{j(\omega_k t + \varphi_k)}$ and the noiseless data matrix

$$\mathbf{X} = \begin{bmatrix} x(1) & x(L+1) & \cdots & x(L(M-1)+1) \\ x(2) & x(L+2) & \cdots & x(L(M-1)+2) \\ \vdots & \vdots & & \vdots \\ x(L) & x(2L) & \cdots & x(LM) \end{bmatrix}. \quad (3)$$

It is easily showed that $E\{q(t)\} = 0$, where $E\{\bullet\}$ denotes the expectation operator.

We observe that $\mathbf{X}$ can be factorized as

$$\mathbf{X} = \mathbf{G\Lambda H^T}, \quad (4)$$

where

$$\begin{aligned}\mathbf{G} &= \begin{bmatrix} e^{j\omega_1} & e^{j\omega_2} & \cdots & e^{j\omega_p} \\ (e^{j\omega_1})^2 & (e^{j\omega_2})^2 & \cdots & (e^{j\omega_p})^2 \\ \vdots & \vdots & & \vdots \\ (e^{j\omega_1})^L & (e^{j\omega_2})^L & \cdots & (e^{j\omega_p})^L \end{bmatrix}, \\ &= [\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_p]\end{aligned} \quad (5)$$

$$\begin{aligned}\mathbf{H} &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{jL\omega_1} & e^{jL\omega_2} & \cdots & e^{jL\omega_p} \\ \vdots & \vdots & & \vdots \\ (e^{jL\omega_1})^{M-1} & (e^{jL\omega_2})^{M-1} & \cdots & (e^{jL\omega_p})^{M-1} \end{bmatrix}, \\ &= [\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_p]\end{aligned} \quad (6)$$

$$\mathbf{\Lambda} = \mathrm{diag}\{\mu_1 e^{j\varphi_1}, \mu_2 e^{j\varphi_2}, \cdots, \mu_p e^{j\varphi_p}\}, \quad (7)$$

and $(\cdot)^T$ denotes the transpose. It can be observed that the frequency information is contained in $\mathbf{G}$ and $\mathbf{H}$ but the frequency estimation is not directly available from $\{y(t)\}$. In this paper, we use the PUMA method [13-16] for frequency estimation as follows.

Let the singular value decomposition (SVD) of the matrix $\mathbf{X}$ be

$$\mathbf{X} = \mathbf{UDV}^H = \mathbf{U}_s\mathbf{D}_s\mathbf{V}_s^H + \mathbf{U}_n\mathbf{D}_n\mathbf{V}_n^H. \quad (8)$$

Here, $(\cdot)^H$ denotes the complex conjugate transpose. $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_L]$ and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_M]$ are two unitary matrices, $\mathbf{D}$ indicates the singular value matrix in which each diagonal element represents a singular value and all entries are arranged in a non-increasing order. $\mathbf{U}_s = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p]$, $\mathbf{V}_s = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p]$ and $\mathbf{D}_s = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$ contain p principal components, i.e., principal left singular vectors, principle right singular vectors and principle singular values of $\mathbf{X}$, and $\mathbf{U}_n$, $\mathbf{D}_n$ and $\mathbf{V}_n$ contain remaining components.

From (4) and (8), we observe that $\mathbf{G}$ and $\mathbf{U}_s$ span the same subspace, namely, there must exist a $p \times p$ non-singular matrix $\mathbf{\Gamma} = (\zeta_{ij})$ such that

$$\mathbf{U}_s = \mathbf{G\Gamma}. \quad (9)$$

Note that in (9), each column of $\mathbf{U}_s$, namely, $\mathbf{u}_k$, $k = 1, 2, \cdots, p$, can be expressed as

$$\begin{aligned}\mathbf{u}_k &= \mathbf{g}_1\zeta_{1k} + \mathbf{g}_2\zeta_{2k} + \cdots + \mathbf{g}_p\zeta_{pk} \\ &= \begin{bmatrix} \zeta_{1k}e^{j\omega_1} + \zeta_{2k}e^{j\omega_2} + \cdots + \zeta_{pk}e^{j\omega_p} \\ \zeta_{1k}(e^{j\omega_1})^2 + \zeta_{2k}(e^{j\omega_2})^2 + \cdots + \zeta_{pk}(e^{j\omega_p})^2 \\ \vdots \\ \zeta_{1k}(e^{j\omega_1})^L + \zeta_{2k}(e^{j\omega_2})^L + \cdots + \zeta_{pk}(e^{j\omega_p})^L \end{bmatrix}. \quad (10)\end{aligned}$$

From (10), $\left[\mathbf{u}_k\right]_l$ can be expressed as [17]

$$[\mathbf{u}_k]_l = -c_1[\mathbf{u}_k]_{l-1} - c_2[\mathbf{u}_k]_{l-2} - \cdots - c_p[\mathbf{u}_k]_{l-p} \\ k = 1,2,\cdots,p, \quad l = p+1,\cdots,L \quad (11)$$

where $\left[\mathbf{a}\right]_i$ denotes the i th element of $\mathbf{a}$. By simple calculations it can be shown that the following p -degree polynomial

$$z^p + c_1 z^{p-1} + \cdots + c_{p-1} z + c_p = 0 \quad (12)$$

has roots $e^{j\omega_1}, e^{j\omega_2}, \cdots, e^{j\omega_p}$. Here it indicates that when $c_1, c_2, \cdots, c_p$ are estimated, $\omega_1, \omega_2, \cdots, \omega_p$ can be estimated.

We write (11) as following matrix form

$$\begin{bmatrix} [\mathbf{u}_k]_{p+1} & [\mathbf{u}_k]_p & \cdots & [\mathbf{u}_k]_1 \\ [\mathbf{u}_k]_{p+2} & [\mathbf{u}_k]_{p+1} & \cdots & [\mathbf{u}_k]_2 \\ \vdots & \vdots & & \vdots \\ [\mathbf{u}_k]_L & [\mathbf{u}_k]_{L-1} & \cdots & [\mathbf{u}_k]_{L-p} \end{bmatrix} \begin{bmatrix} 1 \\ c_1 \\ \vdots \\ c_p \end{bmatrix}, \quad (13)$$
$$= \mathbf{A}_k\mathbf{c} - \mathbf{b}_k = \mathbf{C}\mathbf{u}_k = \mathbf{0}; \qquad k = 1,\cdots,p$$

where

$$\mathbf{A}_k = \begin{bmatrix} [\mathbf{u}_k]_p & [\mathbf{u}_k]_{p-1} & \cdots & [\mathbf{u}_k]_1 \\ [\mathbf{u}_k]_{p+1} & [\mathbf{u}_k]_p & \cdots & [\mathbf{u}_k]_2 \\ \vdots & \vdots & & \vdots \\ [\mathbf{u}_k]_{L-1} & [\mathbf{u}_k]_{L-2} & \cdots & [\mathbf{u}_k]_{L-p} \end{bmatrix}, \quad (14)$$

$$\mathbf{b}_k = -\left[[\mathbf{u}_k]_{p+1}, [\mathbf{u}_k]_{p+2}, \cdots, [\mathbf{u}_k]_L\right]^T, \quad (15)$$

$$\mathbf{c} = [c_1, c_2, \cdots, c_p]^T, \quad (16)$$

and

$$\mathbf{C} = \begin{bmatrix} c_p & c_{p-1} & c_{p-2} & c_{p-3} & \cdots & c_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_p & c_{p-1} & c_{p-2} & \cdots & c_2 & c_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_p & c_{p-1} & \cdots & c_1 & 1 \end{bmatrix}_{(L-p)\times L} \quad (17)$$

Putting all of these ((13) for $k = 1,\cdots,p$) together, we find that

$$\mathbf{A}\mathbf{c} - \mathbf{b} = \begin{bmatrix} \mathbf{A}_1\mathbf{c} - \mathbf{b}_1 \\ \mathbf{A}_2\mathbf{c} - \mathbf{b}_2 \\ \vdots \\ \mathbf{A}_p\mathbf{c} - \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{C}\mathbf{u}_1 \\ \mathbf{C}\mathbf{u}_2 \\ \vdots \\ \mathbf{C}\mathbf{u}_p \end{bmatrix} = \mathrm{vec}(\mathbf{C}\mathbf{U}_s) = \mathbf{0}, \quad (18)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_p \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_p \end{bmatrix}. \quad (19)$$

In order to obtain the estimation of $\mathbf{c}$, we need exploit the SVD of noise data matrix $\tilde{\mathbf{X}}$, which is given by

$$\tilde{\mathbf{X}} = \mathbf{X} + \mathbf{Q} = \tilde{\mathbf{U}}\tilde{\mathbf{D}}\tilde{\mathbf{V}} = \tilde{\mathbf{U}}_s\tilde{\mathbf{D}}_s\tilde{\mathbf{V}}_s^H + \tilde{\mathbf{U}}_n\tilde{\mathbf{D}}_n\tilde{\mathbf{V}}_n^H, \quad (20)$$

where $\tilde{\mathbf{U}} = [\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \cdots, \tilde{\mathbf{u}}_L]$ and $\tilde{\mathbf{V}} = [\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \cdots, \tilde{\mathbf{v}}_M]$ are two unitary matrices, and $\tilde{\mathbf{D}}$ indicates the singular value matrix in which each diagonal element represents a singular value and all entries are arranged in a non-increasing order, $\tilde{\mathbf{U}}_s = [\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \cdots, \tilde{\mathbf{u}}_p]$, $\tilde{\mathbf{V}}_s = [\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \cdots, \tilde{\mathbf{v}}_p]$, and $\tilde{\mathbf{D}}_s = \mathrm{diag}\{\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_p\}$ contain the p principal components, i.e., the principal left singular vectors, principle right singular vectors and principle singular values of $\tilde{\mathbf{X}}$, and $\tilde{\mathbf{U}}_n$, $\tilde{\mathbf{D}}_n$ and $\tilde{\mathbf{V}}_n$ contain the remaining components. Let $\tilde{\mathbf{U}}_s = \mathbf{U}_s + \Delta\mathbf{U}_s$ and $\tilde{\mathbf{V}}_s = \mathbf{V}_s + \Delta\mathbf{V}_s$, where $\Delta\mathbf{X}$ is the perturbation of $\mathbf{X}$, respectively.

Now consider the expression $\tilde{\mathbf{A}}\mathbf{c} - \tilde{\mathbf{b}}$. As the perturbation $\Delta\mathbf{U}_s$ [19] is $\Delta\mathbf{U}_s = \mathbf{U}_n\mathbf{U}_n^H\mathbf{Q}\mathbf{V}_s\mathbf{D}_s^{-1}$, we can deduce $E\{\Delta\mathbf{U}_s\} = 0$ and

$$E\{\tilde{\mathbf{A}}\mathbf{c} - \tilde{\mathbf{b}}\} = E\{\mathrm{vec}(\mathbf{C}\tilde{\mathbf{U}}_s\} \\ = E\{\mathrm{vec}(\mathbf{C}(\mathbf{U}_s + \Delta\mathbf{U}_s))\} = E\{\mathrm{vec}(\mathbf{C}\Delta\mathbf{U}_s)\} = 0 \quad (21)$$

From (21), we get

$$\tilde{\mathbf{A}}\mathbf{c} \approx \tilde{\mathbf{b}}. \quad (22)$$

The WLS solution of (22) is given by [20]

$$\hat{\mathbf{c}}_{WLS} \approx (\tilde{\mathbf{A}}^H\mathbf{W}\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^H\mathbf{W}\tilde{\mathbf{b}}. \quad (23)$$

where $\hat{\mathbf{a}}$ denotes the estimation of $\mathbf{a}$ and $\mathbf{W}$ is a weighting matrix. The optimum weighting matrix $\mathbf{W}$ can be expressed as [20]

$$\mathbf{W} = [E\{(\tilde{\mathbf{A}}\mathbf{c} - \tilde{\mathbf{b}})(\tilde{\mathbf{A}}\mathbf{c} - \tilde{\mathbf{b}})^H\}]^{-1} \\ = [E\{\mathrm{vec}(\mathbf{C}\Delta\mathbf{U}_s)(\mathrm{vec}(\mathbf{C}\Delta\mathbf{U}_s)^H\}]^{-1}. \quad (24)$$

Based on $\Delta\mathbf{U}_s = \mathbf{U}_n\mathbf{U}_n^H\mathbf{Q}\mathbf{V}_s\mathbf{D}_s^{-1}$, we have

$$E\{\mathrm{vec}(\mathbf{C}\Delta\mathbf{U}_s)(\mathrm{vec}(\mathbf{C}\Delta\mathbf{U}_s)^H\} \\ = E\{[(\mathbf{D}_s^{-1}\mathbf{V}_s^T \otimes \mathbf{C}\mathbf{U}_n\mathbf{U}_n^H)\mathrm{vec}(\mathbf{Q})] \cdot \\ [(\mathbf{D}_s^{-1}\mathbf{V}_s^T \otimes \mathbf{U}_n\mathbf{U}_n^H\mathbf{C}^H)\mathrm{vec}(\mathbf{Q})]^H\} \\ = (\mathbf{D}_s^{-1}\mathbf{V}_s^T \otimes \mathbf{C}\mathbf{U}_n\mathbf{U}_n^H)E\{\mathrm{vec}(\mathbf{Q})(\mathrm{vec}(\mathbf{Q}))^H\} \\ (\mathbf{V}_s^*\mathbf{D}_s^{-1} \otimes \mathbf{U}_n\mathbf{U}_n^H\mathbf{C}^H)$$

$$\begin{aligned}&=\sigma^2(\mathbf{D}_s^{-1}\mathbf{V}_s^T\mathbf{V}_s^*\mathbf{D}_s^{-1})\otimes(\mathbf{C}\mathbf{U}_n\mathbf{U}_n^H\mathbf{U}_n\mathbf{U}_n^H\mathbf{C}^H)\\&=\sigma^2\mathbf{D}_s^{-2}\otimes\mathbf{C}\mathbf{U}_n\mathbf{U}_n^H\mathbf{C}^H\\&=\sigma^2\mathbf{D}_s^{-2}\otimes\mathbf{C}(\mathbf{I}_L-\mathbf{U}_s\mathbf{U}_s^H)\mathbf{C}^H\\&=\sigma^2\mathbf{D}_s^{-2}\otimes\mathbf{C}\mathbf{C}^H\end{aligned}, \quad (25)$$

where $\otimes$ denotes the Kronecker product, $\sigma^2=E\{q(t)(q(t))^*\}$ and $(\cdot)^*$ denotes the conjugate.

Thus,

$$\mathbf{W}=\sigma^{-2}\mathbf{D}_s^{\,2}\otimes(\mathbf{C}\mathbf{C}^H)^{-1}$$
$$=\sigma^{-2}\begin{bmatrix}\lambda_1^2(\mathbf{C}\mathbf{C}^H)^{-1} & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \lambda_2^2(\mathbf{C}\mathbf{C}^H)^{-1} & \cdots & \mathbf{O}\\ \vdots & \vdots & & \vdots\\ \mathbf{O} & \cdots & \mathbf{O} & \lambda_p^2(\mathbf{C}\mathbf{C}^H)^{-1}\end{bmatrix} \quad (26)$$

It can be observed that the value σ^2 can be canceled out from (23), thus $\hat{\mathbf{c}}$ can be represented as

$$\hat{\mathbf{c}}\approx(\sum_{k=1}^{p}\lambda_k^2\tilde{\mathbf{A}}_k^H(\mathbf{C}\mathbf{C}^H)^{-1}\tilde{\mathbf{A}}_k)^{-1}(\sum_{k=1}^{p}\lambda_k^2\tilde{\mathbf{A}}_k^H(\mathbf{C}\mathbf{C}^H)^{-1}\tilde{\mathbf{b}}_k) \quad (27)$$

Since λ_k is not available from the observed data, we will substitute λ_k with $\tilde{\lambda}_k$ in (27) thus,

$$\hat{\mathbf{c}}\approx(\sum_{k=1}^{p}\tilde{\lambda}_k^2\tilde{\mathbf{A}}_k^H(\mathbf{C}\mathbf{C}^H)^{-1}\tilde{\mathbf{A}}_k)^{-1}(\sum_{k=1}^{p}\tilde{\lambda}_k^2\tilde{\mathbf{A}}_k^H(\mathbf{C}\mathbf{C}^H)^{-1}\tilde{\mathbf{b}}_k). \quad (28)$$

The estimation procedure based on the principal left singular vectors $\tilde{\mathbf{U}}_s$ of $\tilde{\mathbf{X}}$ is summarized as follows.

Step1: Use the observed signal to construct a matrix $\tilde{\mathbf{X}}$ by (1) and (2).

Step2: Compute the two matrices $\tilde{\mathbf{U}}_s$ and $\tilde{\mathbf{D}}_s$ of (20) by performing the singular value decomposition on $\tilde{\mathbf{X}}$.

Step3: Build $\tilde{\mathbf{A}}_k$ of (14) and $\tilde{\mathbf{b}}_k$ of (15) by using each column of $\tilde{\mathbf{U}}_s$, namely, $\mathbf{u}_k$, and build $\tilde{\lambda}_k$ by using $\tilde{\mathbf{D}}_s$.

Step4: Set $\mathbf{C}\mathbf{C}^H=\mathbf{I}_{L-p}$.

Step5: Compute $\hat{\mathbf{c}}$ by using (28).

Step6: Compute updated $\mathbf{C}$ by using (17).

Step7: Iterate steps 5-6 until a stopping criterion is reached.

Step8: Substitute $\hat{\mathbf{c}}=[c_1,c_2,\cdots,c_p]^T$ in (12) and solve for the roots

$$\{\hat{a}_k;\ \ k=1,2,\cdots,p\}. \quad (29)$$

Step9: Estimate the frequencies as follows:

$$\{\hat{\omega}_{L,k}=\angle(\hat{a}_k);\ \ k=1,2,\cdots,p\}, \quad (30)$$

where $\angle$ is the phase angle operator.

Basically, we can use similar manner above to solve for $\hat{\omega}_{R,k}$ (let $\hat{\omega}_{R,k}=L\hat{\omega}_k$) obtained from the principal right singular vectors $\tilde{\mathbf{V}}_s$ of $\tilde{\mathbf{X}}$. However, $\hat{\omega}_{R,k}$ corresponds to $2\lfloor L/2\rfloor+1$ possible estimates of $\hat{\omega}_k$, where $\lfloor a\rfloor$ denotes the maximum integer not exceeding a, denoted by $\hat{\omega}_{R,k,i}$, $i=-\lfloor L/2\rfloor,-\lfloor L/2\rfloor+1,\cdots,\lfloor L/2\rfloor$:

$$\hat{\omega}_{R,k,i}=\frac{\hat{\omega}_{R,k}+2\pi i}{L}. \quad (31)$$

A simple way of finding $\hat{\omega}_k$ from $\{\hat{\omega}_{R,k,i}\}$ is to compare each of them with $\hat{\omega}_{L,k}$, that is, the estimation of frequencies based on the principal right singular vectors $\tilde{\mathbf{V}}_s$ of $\tilde{\mathbf{X}}$ is given by $\hat{\omega}_{R,k,i^*}$, where i^* is obtained from

$$i^*=\arg\min_{i\in\{-\lfloor L/2\rfloor,-\lfloor L/2\rfloor+1,\cdots,\lfloor L/2\rfloor\}}\left|\hat{\omega}_{R,k,i}-\hat{\omega}_{L,k}\right|. \quad (32)$$

However, this is required to determine the correct pairs of $(\hat{\omega}_{R,k},\hat{\omega}_{L,k})$. We follow [14] to achieve frequency pairing $(\hat{\omega}_{R,k},\hat{\omega}_{L,k})$ in an automatic manner as follows.

From (2) and (4), we get

$$\tilde{\mathbf{X}}\approx\hat{\mathbf{G}}\mathbf{F}^T. \quad (33)$$

Here, $\hat{\mathbf{G}}$ is constructed according to

$$\hat{\mathbf{G}}=[\hat{\mathbf{g}}_1,\hat{\mathbf{g}}_2,\cdots,\hat{\mathbf{g}}_p], \quad (34)$$

and $\{\hat{\mathbf{g}}_k\}$ is obtained from $\{\hat{a}_k\}$, $\mathbf{F}^T=\mathbf{\Lambda}\mathbf{H}^T$ and

$$\begin{aligned}\mathbf{F}&=[\mathbf{f}_1,\mathbf{f}_2,\cdots,\mathbf{f}_p]\\&=[\mu_1e^{j\varphi_1}\mathbf{h}_1,\mu_2e^{j\varphi_2}\mathbf{h}_2,\cdots,\mu_pe^{j\varphi_p}\mathbf{h}_p]\end{aligned}. \quad (35)$$

From (33), the least square (LS) estimate of $\mathbf{F}$ is

$$\hat{\mathbf{F}}\approx\mathbf{X}^T(\hat{\mathbf{G}}^\dagger)^T. \quad (36)$$

We use the notation $\overline{\mathbf{X}}$ and $\underline{\mathbf{X}}$ to denote the matrix $\mathbf{X}$ with the first and last row omitted, respectively.

From (35), we have

$$\underline{\hat{\mathbf{f}}}_k b_k=\overline{\hat{\mathbf{f}}}_k \quad (37)$$

where $\{b_k=e^{jL\omega_k}=e^{j\omega_{R,k}}\}$. Following [18], the WLS estimate of b_k is computed as:

$$\hat{b}_k\approx(\underline{\hat{\mathbf{f}}}_k^H\mathbf{Z}_k\underline{\hat{\mathbf{f}}}_k)^{-1}\underline{\hat{\mathbf{f}}}_k^H\mathbf{Z}_k\overline{\hat{\mathbf{f}}}_k, \quad (38)$$

$$\mathbf{Z}_k = (\mathbf{B}_k \mathbf{B}_k^H)^{-1}, \quad (39)$$

and

$$\mathbf{B}_k = \begin{pmatrix} b_k & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_k & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & b_k & -1 \end{pmatrix}_{(M-1)\times M}. \quad (40)$$

Finally, the estimate of $\hat{\omega}_{R,k}$ is

$$\hat{\omega}_{R,k} = \angle(b_k); \ \ k = 1, 2, \cdots, p. \quad (41)$$

The estimation procedure based on the principal right singular vectors $\tilde{\mathbf{V}}_s$ of $\tilde{\mathbf{X}}$ is summarized as follows.

Step1: Use (29), (33), (34) and (36) to obtain the matrix $\hat{\mathbf{F}}$.

Step2: Build the matrices $\underline{\hat{\mathbf{f}}}_k$, and $\overline{\hat{\mathbf{f}}}_k$ by using $\hat{\mathbf{F}}$.

Step3: Set $\mathbf{B}_k \mathbf{B}_k^H = \mathbf{I}_{M-1}$.

Step4: Compute $\hat{b}_k$ by using (38).

Step5: Compute updated $\mathbf{B}_k$ by using (40).

Step6: Iterate steps 4-5 until a stopping criterion is reached.

Step7: Compute $\hat{\omega}_{R,k} = \angle(b_k); \ k = 1, 2, \cdots, p$.

Step8: Determine $\hat{\omega}_{R,k,i^*}$ from $\hat{\omega}_{R,k}$ according to (31)-(32).

It will be shown in Section 3 that using $\hat{\omega}_{R,k,i^*}$ has a much higher accuracy that that of $\hat{\omega}_{L,k}$. Therefore, $\hat{\omega}_{R,k,i^*}$ is considered as the final estimates.

III. Numerical Experiments

In this section, we present some experimental results to see how the proposed method behaves for finite samples. We consider the following model:

$$y(t) = s_1(t)e^{j(0.3t+0.1)} + s_2(t)e^{j(1.2t+0.2)} + v(t); \quad t = 1, 2, \cdots, N \quad . \quad (42)$$

We assume that $\{v(t)\}$ is taken as a sequence of i.i.d. Gaussian complex random variable with zero-mean and finite variance σ_v^2. We consider $\{s_1(t)\}$ and $\{s_2(t)\}$ to be i.i.d. Gaussian real random variables with means 2, 2 and deviations 0.2, 0.3 respectively. We want to see how the proposed subspace method behaves for different noise levels and for different sample sizes. We consider σ_v=0.5, 1, 1.5 and $N = 256$, 1024. In all cases, we use three iterations because no significant improvement is observed for more iterations. We compute the mean estimations (ME) and the mean square errors (MSEs) of the frequency estimates of model (42) over 1000 simulation runs based on a computer with Intel Core 2.67 GHz processors and 3.25 GB RAM and the results are presented in Tables 1-3. In each table the first row in each of the cell represents the true frequency values, the corresponding ME and MSEs are reported in the second and last rows, respectively.

Tables 1-2 show the performance of the proposed method based on $\hat{\omega}_{L,k}$ and $\hat{\omega}_{R,k,i^*}$ at $N = 256$ with different combinations of *L* and *M*, respectively. It is very clear from Tables 1-2 that $(L, M) = (8, 32)$, $(32, 8)$ and $(16,16)$ are best choices. It is also observed that the biases increase as the additive noise deviation increases, which indicates that the proposed method for larger σ_v it is more difficult to estimate the unknown frequencies. Compare Table 1 with 2, for different values of σ_v, $\hat{\omega}_{R,k,i^*}$ works better than $\hat{\omega}_{L,k}$. Moreover, the results of the simulation experiments show that the proposed subspace method based on $\hat{\omega}_{R,k,i^*}$ is also fairly good even when σ_v is large. For $N = 256$, the average computation times of the proposed estimator based on $\hat{\omega}_{R,k,i^*}$ with $(L, M) = (4, 64)$, $(8, 32)$, $(16,16)$, $(32, 8)$ and $(64, 4)$ are measured as 9.12×10^{-3} s, 3.42×10^{-3} s, 9.12×10^{-3} s, 4.67×10^{-3} s and 1.32×10^{-2} s, respectively. Summarizing the results, $\hat{\omega}_{R,k,i^*}$ is considered as the best estimates and the best combination in terms of accuracy and computational complexity is $L \approx M$.

In table 3, we report the ME and MSEs when the additive noise deviation is 1, and the sample sizes $N = 256$ and $N = 1024$ with $(L, M) = (16,16)$ and $(L, M) = (32, 32)$, respectively. It is observed that as the sample size N increases the MSEs decrease, it verifies the consistency property of the proposed method for the frequency estimation.

TABLE III.

The ME and MSEs of frequencies with N = 256, 1024 and σ_v =1

σ_v	Estimate	$(L,M)=(16,16)$		$(L,M)=(32,32)$	
1	Parameter	0.3000	1.2000	0.3000	1.2000
	ME	0.3000	1.2000	0.3000	1.2000
	MSEs	2.88e-4	2.49e-4	6.97e-5	6.16e-5

IV. Conclusions

In this paper, we considered the estimation of frequencies of a superimposed exponential signal model. We generalized the PUMA method [13-16] from sinusoids with additive noise to multiple signals with multiplicative and additive noise. The techniques SVD and WLS are used to obtain the frequency estimation. Computer simulations show that the proposed subspace method is computationally attractive and work well in the case of long sample size and/or small noise deviation.

TABLE I.

THE ME AND MSES OF FREQUENCIES WITH $N = 256$ AND $\hat{\omega}_{L,k}$

σ_v	Estimate	$(L,M)=(4,64)$		$(L,M)=(8,32)$		$(L,M)=(16,16)$		$(L,M)=(32,8)$		$(L,M)=(64,4)$	
0.1	Parameter	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	ME	0.2993	1.2003	0.2997	1.2002	0.2999	1.2000	0.3000	1.2000	0.3000	1.2000
	MSEs	1.97e-2	2.01e-2	3.83e-3	2.61e-3	1.64e-3	9.94e-4	7.94e-4	4.72e-4	4.23e-4	2.41e-4
1	Parameter	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	ME	0.2978	1.2012	0.2995	1.2002	0.3000	1.2003	0.3000	1.2001	0.3000	1.2000
	MSEs	4.80e-2	4.42e-2	1.03e-2	9.31e-3	4.71e-3	4.24e-3	2.30e-3	1.97e-3	1.11e-3	9.80e-4
2	Parameter	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	ME	0.2990	1.2077	0.2993	1.2007	0.2994	1.2005	0.2998	1.2000	0.2988	1.2003
	MSEs	1.02e-1	9.36e-2	2.07e-2	1.89e-2	9.22e-3	8.20e-3	4.56e-3	3.96e-3	2.86e-3	2.21e-3

TABLE II.

THE ME AND MSES OF FREQUENCIES WITH $N = 256$ AND $\hat{\omega}_{R,k,i^*}$

σ_v	Estimate	$(L,M)=(4,64)$		$(L,M)=(8,32)$		$(L,M)=(16,16)$		$(L,M)=(32,8)$		$(L,M)=(64,4)$	
0.1	Parameter	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	ME	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	MSEs	1.37e-4	1.10e-4	1.03e-4	5.86e-5	1.01e-4	5.98e-5	1.05e-4	5.73e-5	1.06e-4	6.16e-5
1	Parameter	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	ME	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.2900	1.2000
	MSEs	3.57e-4	3.21e-4	3.00e-4	2.48e-4	2.88e-4	2.49e-4	3.07e-4	2.51e-4	2.95e-4	2.58e-4
2	Parameter	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	ME	0.3000	1.2001	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000	0.3000	1.2000
	MSEs	7.06e-4	7.48e-4	5.43e-4	4.85e-4	5.79e-4	5.01e-4	5.74e-4	5.08e-4	5.88e-4	5.09e-4

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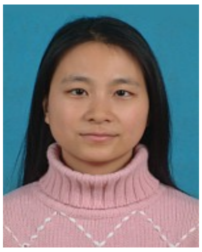
Zhihui Liu is pursuing the Ph.D. degree at China University of Geosciences (CUG). She is currently a lecturer at the School of Mathematics and Physics in CUG. Her research interests include statistical signal processing and time series analysis.



Lihua Fu received the Ph.D. degree in mathematics from China University of Geosciences (CUG). She is also an associate professor with the School of Mathematics and Physics in CUG. Her research interests include wavelet and its applications in signal processing.

Shizhong Zhang is pursuing the Ph.D. degree at China University of Geosciences (CUG). He is currently a lecturer at the School of Mathematics and Physics in CUG. His research interests include signal processing and blind source separation.