The Maximal Relation Based on A Given Relation Schema and Its Concept Lattice

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***Abstract*— Formal concept analysis (FCA) is a valid tool for data mining and knowledge discovery, which identifies concept lattices from binary relations. Given a nonempty finite set A of binary attributes, one obtains a maximal binary relation $R^{\max}$ based on a relation schema $S(A)$. Firstly, we analyze concepts in $R^{\max}$ and the concept lattice $\mathcal{L}(R^{\max})$, and there are some important results as follows: for any two concepts in $R^{\max}$, the union of their intents is an intent of some concept in $R^{\max}$, and further the intent of their supremum is the union of their intents; for any two concepts in $R^{\max}$, if one of them is not a sub-concept or super-concept of the other one, then the union of their extents is not an extent of any concept in $R^{\max}$; $\mathcal{L}(R^{\max})$ is a complemented distributive lattice. Secondly, we provide the structural connection between $\mathcal{L}(R)$ and $\mathcal{L}(R^{\max})$: for any relation R based on $S(A)$, there is a supremum-preserving order-embedding map from $\mathcal{L}(R)$ to $\mathcal{L}(R^{\max})$, and conversely, there is an infimum-preserving order-preserving map from $\mathcal{L}(R^{\max})$ to $\mathcal{L}(R)$, which is generally not a surjective homomorphism. Thirdly, we propose two algorithms to extract concepts in R from $\mathcal{L}(R^{\max})$, which are respectively based on intents and extents of concepts, and prove their soundness. These results have already been used to analyze the data in architectual engineering and medical science.**

***Index Terms*— concept lattices; relation schemas; maximal binary relations; order-preserving maps**

I. INTRODUCTION

FCA[6] is an effective data analysis technique, which automatically generates hierarchies called concept lattices from contexts. Recently, concept lattices have already been successfully applied to a wide range of scientific disciplines such as knowledge discovery [1, 3, 4, 5, 8-11, 15, 18], information retrieval [2, 13, 16, 19], software engineering [12, 20], rough set theory [17, 21, 23, 25], and knowledge ontology [7, 14].

As many practical applications involve binary data, this paper discusses the concept lattices of binary relations (contexts). Given a nonempty finite set A of binary attributes, one obtains a maximal binary relation $R^{\max}$ on a relation schema $S(A)$. This paper mainly analyzes concepts in $R^{\max}$ and the concept lattice of $R^{\max}$. Generally, given a relation R based on $S(A)$, for any two concepts in R, the union of their intents may not be an intent of any concept in R. However, compared with general relations, $R^{\max}$ is a special one, and thereby it has more properties, and this paper obtains the following results: for any two concepts in $R^{\max}$, the union of their intents is an intent of some concept in $R^{\max}$, and further the intent of their supremum is the union of their intents; for any two concepts in $R^{\max}$, if one of them is not a sub-concept or super-concept of the other one, then the union of their extents is not an extent of any concept in $R^{\max}$; $\mathcal{L}(R^{\max})$ is a complemented distributive lattice. The connection among three concepts in $R^{\max}$ is also considered. For any relation R based on $S(A)$, R is a sub-relation of $R^{\max}$, and their concept lattices have the following structural connection: there is a supremum-preserving order-embedding map from $\mathcal{L}(R)$ to $\mathcal{L}(R^{\max})$, and conversely, there is an infimum-preserving order-preserving map from $\mathcal{L}(R^{\max})$ to $\mathcal{L}(R)$, which is generally not a surjective homomorphism. Furthermore, this paper provides two equivalent algorithms to extract all concepts in R from $\mathcal{L}(R^{\max})$.

This paper is organized as follows. Section 2 gives some necessary notions. Section 3 discusses concepts in $R^{\max}$ and the concept lattice of $R^{\max}$. Section 4 firstly analyzes the connection between the concept lattice of R and the concept lattice of $R^{\max}$, and secondly provides two algorithms to extract concepts in R from the concept lattice of $R^{\max}$. Section 5 concludes the paper.

II. PRELIMINARIES

In FCA, a *context* $K = (G, M, I)$ consists of two sets G and M and a relation I between G and M. The elements of G and M are respectively called objects and attributes. For any $g \in G$ and $m \in M$, $(g, m) \in I$ (or gIm) implies that the object g possesses the attribute m [6]. The relation I induces two maps f^K and h^K between the power set $\mathcal{P}(G)$ of G and the power set $\mathcal{P}(M)$ of M. For a set $X \in \mathcal{P}(G)$ of objects, $f^K(X)$ is defined as:

$$f^K(X) = \{m \in M : \forall g \in X(gIm)\},$$

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which is the set of attributes common to the objects in X. Correspondingly, for a set $Y \in \mathcal{P}(M)$ of attributes, $h^K(Y)$ is defined as:

$$h^K(Y) = \{g \in G : \forall m \in Y(gIm)\},$$

which is the set of objects which have all attributes in Y.

Given a context $K = (G, M, I)$, for any $X \in \mathcal{P}(G)$ and $Y \in \mathcal{P}(M)$, the pair (X, Y) is called a *(formal) concept* if $f^K(X) = Y$ and $h^K(Y) = X$, where X and Y are respectively called the *extent* and the *intent* of the concept. The set of all concepts of K is denoted by $\mathcal{L}(K)$. If $(X_1, Y_1), (X_2, Y_2) \in \mathcal{L}(K)$ are concepts, (X_1, Y_1) is called a *sub-concept* of (X_2, Y_2), provided that $X_1 \subseteq X_2$ (which is equivalent to $Y_2 \subseteq Y_1$), denoted by $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$. In this case, (X_2, Y_2) is a *super-concept* of (X_1, Y_1). The relation $\sqsubseteq$ is an order on $\mathcal{L}(K)$, called the *hierarchical order* of the concepts. The hierarchical order produces a lattice structure in $\mathcal{L}(K)$, called the *concept lattice* of the context K, also denoted by $\mathcal{L}(K)$. $\mathcal{L}(K)$ is also a complete lattice in which *infimum* and *supremum* are given by [6]:

$$\bigwedge_{t \in T}(A_t, B_t) = (\bigcap_{t \in T} A_t,\ f^K h^K(\bigcup_{t \in T} B_t)),$$

$$\bigvee_{t \in T}(A_t, B_t) = (h^K f^K(\bigcup_{t \in T} A_t),\ \bigcap_{t \in T} B_t),$$

where T is an index set, and infimum and supremum respectively represent the *largest common sub-concept* and the *small common super-concept* of some concepts.

Actually, a binary relation in relation databases is a context in FCA. A binary relation R based on a relation schema $S(A)$ can be represented by a triple (U, A, I), where U is a nonempty set of tuples, A is a set of binary attributes, and I is a map from $U \times A$ to $\{0, 1\}$ such that for any $(r, a) \in U \times A$, $I(r, a) = 1 \Leftrightarrow rIa$. In the following sections, we write $r(a) = 1$ instead of $I(r, a) = 1$, and use $\sqcap$ and $\sqcup$ instead of $\bigwedge$ and $\bigvee$, respectively. For every concept in a relation, the extent and the intent of the concept are closely connected by the map I, and each of the parts determines the other and thereby the concept. The next descriptions state further rules of this interaction: Given a relation $R = (U, A, I)$, $X, X_1, X_2 \subseteq U$ are sets of tuples, then

- $X_1 \subseteq X_2 \Rightarrow f^R(X_2) \subseteq f^R(X_1)$
- $X \subseteq h^R f^R(X)$
- $f^R(X) = f^R h^R f^R(X)$
- $X \subseteq h^R(Y) \Leftrightarrow Y \subseteq f^R(X)$
- $f^R(X_1 \cup X_2) = f^R(X_1) \cap f^R(X_2)$
- $f^R(X_1) \cup f^R(X_2) \subseteq f^R(X_1 \cap X_2)$
- $X_1 \subseteq X_2 \Rightarrow h^R f^R(X_1) \subseteq h^R f^R(X_2)$.

Dually, $Y, Y_1, Y_2 \subseteq A$ are sets of attributes, then

- $Y_1 \subseteq Y_2 \Rightarrow h^R(Y_2) \subseteq h^R(Y_1)$
- $Y \subseteq f^R h^R(Y)$
- $h^R(Y) = h^R f^R h^R(Y)$
- $h^R(Y_1 \cup Y_2) = h^R(Y_1) \cap h^R(Y_2)$
- $h^R(Y_1) \cup h^R(Y_2) \subseteq h^R(Y_1 \cap Y_2)$
- $Y_1 \subseteq Y_2 \Rightarrow f^R h^R(Y_1) \subseteq f^R h^R(Y_2)$.

III. The Maximal Relation Based on A Given Relation Schema and Its Concept Lattice

Given a nonempty finite set A of binary attributes, we obtain a maximal relation $R^{\max} = (U^{\max}, A, I^{\max})$, which is based on the schema $S(A)$. The maximality of $R^{\max}$ can be described as follows: for any attribute $a \in A$, there are two tuples $r_1, r_2 \in U^{\max}$ such that $r_1(a) = 1$ and $r_2(a) = 0$, and for any relation $R = (U, A, I)$, there is $U \subseteq U^{\max}$. For any $X \subseteq U^{\max}$ and $Y \subseteq A$, the pair (X, Y) is a concept in $R^{\max}$, if there are:

$$f^{R^{\max}}(X) = \{a \in A : \forall r \in X(r(a) = 1)\} = Y,$$
$$h^{R^{\max}}(Y) = \{r \in U^{\max} : \forall a \in Y(r(a) = 1)\} = X.$$

The concept lattice of $R^{\max}$ is denoted by $\mathcal{L}(R^{\max})$. By the maximality of $R^{\max}$, there is a tuple, which has all attributes in A, and therefore the extent of the smallest concept in $R^{\max}$ is not empty, i.e., $h^{R^{\max}}(A) \neq \emptyset$. Obviously, for any two concept in $R^{\max}$, the intersection of their extents is not empty.

Generally, given a binary relation $R = (U, A, I)$, a subset of A may not be an intent of any concept in R. However, for $R^{\max}$, each subset of A is an intent of some concept in $R^{\max}$. In other words, for any nonempty subset $Y \subseteq A$ and attribute $a \notin Y$, there exists a tuple $r \in U^{\max}$, which has all attributes in Y, but does not have a.

Proposition 3.1. For every set $Y \subseteq A$, Y is an intent of some concept in $R^{\max}$. Obviously, there are $2^{|A|}$ concepts in $R^{\max}$, where $2^{|A|}$ is the number of all subsets of A.

Proof: For any $Y \subseteq A$, we must show that $f^{R^{\max}} h^{R^{\max}}(Y) = Y$. There are the following cases:

Case 1: if $Y = \emptyset$, then the proposition holds.

Case 2: if $Y = A$, then the proposition holds.

Case 3: if $Y \neq \emptyset, A$. We have that $Y \subseteq f^{R^{\max}} h^{R^{\max}}(Y)$. Conversely, assume that there exists an attribute $a \in f^{R^{\max}} h^{R^{\max}}(Y)$ but $a \notin Y$. As $a \in f^{R^{\max}} h^{R^{\max}}(Y)$, we have that:

$$\begin{aligned} & a \in f^{R^{\max}} h^{R^{\max}}(Y) \\ \Leftrightarrow\ & \forall r \in h^{R^{\max}}(Y)(r(a) = 1) \\ \Leftrightarrow\ & \forall r(\forall b \in Y(r(b) = 1) \rightarrow r(a) = 1). \end{aligned}$$

This means that for any tuple $r \in U^{\max}$, if r has all attributes in Y, then r also has the attribute a. However, the result does not hold in $R^{\max}$. In fact, we can construct a tuple r' as follows: for any attribute $b \in A$,

$$r'(b) = \begin{cases} 1 & \text{if } b \in Y \\ 0 & \text{if } b = a \\ 0 & \text{else} \end{cases}$$

Obviously, $r' \in U^{\max}$ has all attributes in Y, but r' does not have the attribute a. This leads to a contradiction. Therefore, Y is an intent of some concept in $R^{\max}$, i.e., each subset of A is an intent of some concept, and hence there is $2^{|A|}$ concepts in $R^{\max}$.

Example 3.1. Given a set $A = \{a, b, c\}$ of binary attributes, based on the schema $S(A)$, we obtain the maximal binary relation $R^{\max}$ as follows:

$$R^{\max} = \begin{array}{c|ccc} & a & b & c \\ \hline r_1 & 1 & 1 & 1 \\ r_2 & 1 & 1 & 0 \\ r_3 & 1 & 0 & 1 \\ r_4 & 1 & 0 & 0 \\ r_5 & 0 & 1 & 1 \\ r_6 & 0 & 1 & 0 \\ r_7 & 0 & 0 & 1 \\ r_8 & 0 & 0 & 0 \end{array}$$

The concept lattice of $R^{\max}$ is shown in Figure 1. In order to make the above proposition more clearer, we label the extents and the intents of all concepts in the concept lattice. From the figure, we easily obtain the following results: for any two concepts, the intersection of their intents is not empty; each subset of A is an intent of some concept, and there are $2^3 = 8$ concepts.

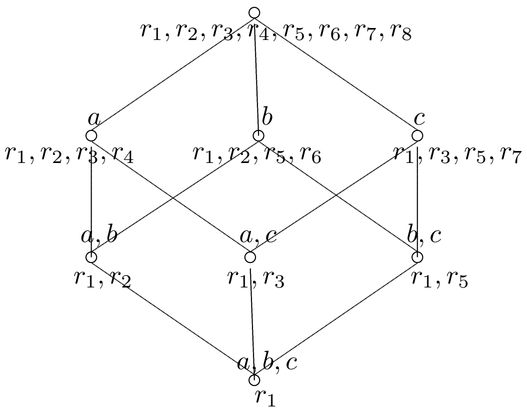


Figure 1. The concept lattice of $R^{\max}$ in example 3.1

About concepts in $R^{\max}$, we have further the following results: for any concepts (X_1, Y_1) and (X_2, Y_2), the set of attributes common to the tuples in $X_1 \cap X_2$ is equivalent to $Y_1 \cup Y_2$, which means that the intent of their supremum is the union of their intents; for any concepts (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3), the set of attributes common to the tuples in $X_1 \cap (X_2 \cup X_3)$ is equivalent to $Y_1 \cup (Y_2 \cap Y_3)$, and the set of attributes common to the tuples in $X_1 \cup (X_2 \cap X_3)$ is equivalent to $Y_1 \cap (Y_2 \cup Y_3)$, as shown in the next proposition.

Proposition 3.2. For any concepts (X_1, Y_1), (X_2, Y_2), $(X_3, Y_3) \in \mathcal{L}(R^{\max})$, there are

- $f^{R^{\max}}(X_1 \cap X_2) = Y_1 \cup Y_2$
- $h^{R^{\max}} f^{R^{\max}}(X_1 \cup (X_2 \cap X_3))$ $= h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_2) \cap h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_3)$
- $f^{R^{\max}}(X_1 \cap (X_2 \cup X_3)) = Y_1 \cup (Y_2 \cap Y_3)$
- $f^{R^{\max}}(X_1 \cup (X_2 \cap X_3)) = Y_1 \cap (Y_2 \cup Y_3)$
- $h^{R^{\max}} f^{R^{\max}}(X_1 \cap (X_2 \cup X_3))$ $= X_1 \cap h^{R^{\max}} f^{R^{\max}}(X_2 \cup X_3)$.

Proof: Firstly, if $Y_1 \cup Y_2 = \emptyset$, then the proposition holds. We assume that $Y_1 \cup Y_2 \neq \emptyset$. For any attribute $a \in Y_1 \cup Y_2$, we have that $a \in Y_1$ or $a \in Y_2$. For any tuple $r \in X_1 \cap X_2$, there are $r \in X_1$ and $r \in X_2$. Because (X_1, Y_1) and (X_2, Y_2) are concepts in $R^{\max}$, there are $r(a) = 1$, and thereby $a \in f^{R^{\max}}(X_1 \cap X_2)$, and consequently $Y_1 \cup Y_2 \subseteq f^{R^{\max}}(X_1 \cap X_2)$. Conversely, $f^{R^{\max}}(X_1 \cap X_2) \subseteq Y_1 \cup Y_2$ also holds. Or else, there exists an attribute $a \in f^{R^{\max}}(X_1 \cap X_2)$ but $a \notin Y_1 \cup Y_2$. We construct a tuple r as follows: for any attribute $b \in A$:

$$r(b) = \begin{cases} 1 & \text{if } b \in Y_1 \cup Y_2 \\ 0 & \text{if } b = a \\ 0 & \text{else} \end{cases}$$

Because (X_1, Y_1) and (X_2, Y_2) are concepts, $r \in X_1$ and $r \in X_2$, and hence $r \in X_1 \cap X_2$. Because $a \in f^{R^{\max}}(X_1 \cap X_2)$, $r(a) = 1$, which is not consist to $r(a) = 0$.

Secondly, by the rules described in section 2, there are

$$\begin{aligned} & h^{R^{\max}} f^{R^{\max}}(X_1 \cup (X_2 \cap X_3)) \\ = \; & h^{R^{\max}}(f^{R^{\max}}(X_1 \cup (X_2 \cap X_3))) \\ = \; & h^{R^{\max}}(f^{R^{\max}}(X_1) \cap f^{R^{\max}}(X_2 \cap X_3)) \\ = \; & h^{R^{\max}}(f^{R^{\max}}(X_1) \cap (f^{R^{\max}}(X_2) \cup f^{R^{\max}}(X_3))) \\ = \; & h^{R^{\max}}(Y_1 \cap (Y_2 \cup Y_3)) \\ = \; & h^{R^{\max}}((Y_1 \cap Y_2) \cup (Y_1 \cap Y_3)) \\ = \; & h^{R^{\max}}(Y_1 \cap Y_2) \cap h^{R^{\max}}(Y_1 \cap Y_3) \\ = \; & h^{R^{\max}}(f^{R^{\max}}(X_1) \cap f^{R^{\max}}(X_2)) \cap \\ & h^{R^{\max}}(f^{R^{\max}}(X_1) \cap f(X_3)) \\ = \; & h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_2) \cap h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_3). \end{aligned}$$

Thirdly, because $X_2 \cup X_3$ may not be an extent, we do not directly use the above results. We show that $f^{R^{\max}}(X_1) \cup (f^{R^{\max}}(X_2) \cap f^{R^{\max}}(X_3)) \subseteq f^{R^{\max}}(X_1 \cap (X_2 \cup X_3))$. Actually, there are

$$\begin{aligned} & X_1 \cap (X_2 \cup X_3) \subseteq X_1 \\ \Rightarrow \; & f^{R^{\max}}(X_1) \subseteq f^{R^{\max}}(X_1 \cap (X_2 \cup X_3)) \\ & X_1 \cap (X_2 \cup X_3) \subseteq X_2 \cup X_3 \\ \Rightarrow \; & f^{R^{\max}}(X_2 \cup X_3) \subseteq f^{R^{\max}}(X_1 \cap (X_2 \cup X_3)), \end{aligned}$$

Hence, there are

$$\begin{aligned} & f^{R^{\max}}(X_1) \cup (f^{R^{\max}}(X_2) \cap f^{R^{\max}}(X_3)) \\ = \; & f^{R^{\max}}(X_1) \cup f^{R^{\max}}(X_2 \cup X_3) \\ \subseteq \; & f^{R^{\max}}(X_1 \cap (X_2 \cup X_3)). \end{aligned}$$

Conversely, $f^{R^{\max}}(X_1 \cap (X_2 \cup X_3)) \subseteq f^{R^{\max}}(X_1) \cup (f^{R^{\max}}(X_2) \cap f^{R^{\max}}(X_3)) = Y_1 \cup (Y_2 \cap Y_3)$ also holds. Or else, there exists an attribute $a \in f^{R^{\max}}(X_1 \cap (X_2 \cup X_3))$ but $a \notin Y_1 \cup (Y_2 \cap Y_3) = (Y_1 \cup Y_2) \cap (Y_1 \cup Y_3)$, and thereby $a \notin Y_1 \cup Y_2$ or $a \notin Y_1 \cup Y_3$. Therefore, there are the following cases:

Case 1: $a \notin Y_1 \cup Y_2$ but $a \in Y_1 \cup Y_3$. We construct a tuple r_1 as follows: for any attribute $b \in A$,

$$r_1(b) = \begin{cases} 1 & \text{if } b \in Y_1 \cup Y_2 \\ 0 & \text{if } b = a \\ 0 & \text{else} \end{cases}$$

Obviously, $r_1 \in X_1 \cap (X_2 \cup X_3)$. However, $r_1(a) = 0$, which leads to a contradiction.

Case 2: $a \notin Y_1 \cup Y_3$ but $a \in Y_1 \cup Y_2$. We construct a tuple r_2 as follows: for any attribute $b \in A$,

$$r_2(b) = \begin{cases} 1 & \text{if } b \in Y_1 \cup Y_3 \\ 0 & \text{if } b = a \\ 0 & \text{else} \end{cases}$$

Obviously, $r_2 \in X_1 \cap (X_2 \cup X_3)$. However, $r_2(a) = 0$, which leads to a contradiction.

Case 3: $a \notin Y_1 \cup Y_2$ and $a \notin Y_1 \cup Y_3$. We have that $a \notin Y_1 \cup Y_2 \cup Y_3$, and construct a tuple r_3 as follows: for any attribute $b \in A$,

$$r_3(b) = \begin{cases} 1 & \text{if } b \in Y_1 \cup Y_2 \cup Y_3 \\ 0 & \text{if } b = a \\ 0 & \text{else} \end{cases}$$

Obviously, $r_3 \in X_1 \cap (X_2 \cup X_3)$. However, $r_3(a) = 0$, which leads to a contradiction.

Fourthly, in fact,

$$\begin{aligned} & f^{R^{\max}}(X_1 \cup (X_2 \cap X_3)) \\ = \ & f^{R^{\max}}(X_1) \cap f^{R^{\max}}(X_2 \cap X_3) \\ = \ & f^{R^{\max}}(X_1) \cap (f^{R^{\max}}(X_2) \cup f^{R^{\max}}(X_3)) \\ = \ & Y_1 \cap (Y_2 \cup Y_3). \end{aligned}$$

Fifthly, by the rules described in section 2, there are

$$\begin{aligned} & X_1 \cap h^{R^{\max}} f^{R^{\max}}(X_2 \cup X_3) \\ = \ & X_1 \cap h^{R^{\max}}(f^{R^{\max}}(X_2) \cap f^{R^{\max}}(X_3)) \\ = \ & h^{R^{\max}}(Y_1) \cap h^{R^{\max}}(Y_2 \cap Y_3) \\ = \ & h^{R^{\max}}(Y_1 \cup (Y_2 \cap Y_3)) \\ = \ & h^{R^{\max}}(f^{R^{\max}}(X_1) \cup (f^{R^{\max}}(X_2) \cap f^{R^{\max}}(X_3))) \\ = \ & h^{R^{\max}} f^{R^{\max}}(X_1 \cap (X_2 \cup X_3)). \end{aligned}$$

Corollary 3.1. For any two concepts $(X_1, Y_1), (X_2, Y_2) \in \mathcal{L}(R^{\max})$, there is $(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cup Y_2)$.

Proof: For any $(X_1, Y_1), (X_2, Y_2) \in \mathcal{L}(R^{\max})$, there are

$$\begin{aligned} & (X_1, Y_1) \sqcap (X_2, Y_2) \\ = \ & (X_1 \cap X_2, f^{R^{\max}} h^{R^{\max}}(Y_1 \cup Y_2)) \\ = \ & (X_1 \cap X_2, f^{R^{\max}}(h^{R^{\max}}(Y_1) \cap h^{R^{\max}}(Y_2))) \\ = \ & (X_1 \cap X_2, f^{R^{\max}}(X_1 \cap X_2)) \\ = \ & (X_1 \cap X_2, f^{R^{\max}}(X_1) \cup f^{R^{\max}}(X_2)) \\ = \ & (X_1 \cap X_2, Y_1 \cup Y_2). \end{aligned}$$

As mentioned above, for every set $Y \subseteq A$, $A - Y$ is an intent of some concept in $R^{\max}$. Thus, for any concept (X, Y) in $R^{\max}$, $(h^{R^{\max}}(A - Y), A - Y)$ is also a concept in $R^{\max}$, which is called the complementary concept of (X, Y), denoted by $\sim (X, Y)$.

Proposition 3.3. $\mathcal{L}(R^{\max})$ is a complemented distributive lattice.

Proof: Firstly, $\mathcal{L}(R^{\max})$ is a complemented lattice. Because each subset of A is an intent, for any concept $(X, Y) \in \mathcal{L}(R^{\max})$, $(h^{R^{\max}}(A - Y), A - Y)$ is the complement concept of (X, Y). It is easily inferred that the infimum and the supremum of a concept and its complementary concept are respectively the largest concept and the smallest concept in $R^{\max}$.

Secondly, $\mathcal{L}(R^{\max})$ is a distributive lattice. On the one hand, for any concepts (X_1, Y_1), (X_2, Y_2), $(X_3, Y_3) \in \mathcal{L}(R^{\max})$, there are

$$\begin{aligned} & (X_1, Y_1) \sqcup ((X_2, Y_2) \sqcap (X_3, Y_3)) \\ = \ & (X_1, Y_1) \sqcup (X_2 \cap X_3, Y_2 \cup Y_3) \text{(By Corollery 3.1)} \\ = \ & (h^{R^{\max}} f^{R^{\max}}(X_1 \cup (X_2 \cap X_3)), Y_1 \cap (Y_2 \cup Y_3)) \\ = \ & (h^{R^{\max}} f^{R^{\max}}(X_1 \cup (X_2 \cap X_3)), (Y_1 \cap Y_2) \cup (Y_1 \cap Y_3)) \\ = \ & (h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_2) \cap h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_3), \\ & (Y_1 \cap Y_2) \cup (Y_1 \cap Y_3)) \\ = \ & (h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_2), Y_1 \cap Y_2) \sqcap \\ & (h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_3), Y_1 \cap Y_3) \\ = \ & ((X_1, Y_1) \sqcup (X_2, Y_2)) \sqcap ((X_1, Y_1) \sqcup (X_3, Y_3)). \end{aligned}$$

On the other hand, for any concepts (X_1, Y_1), (X_2, Y_2), $(X_3, Y_3) \in \mathcal{L}(R^{\max})$, there are

$$\begin{aligned} & (X_1, Y_1) \sqcap ((X_2, Y_2) \sqcup (X_3, Y_3)) \\ = \ & (X_1, Y_1) \sqcap (h^{R^{\max}} f^{R^{\max}}(X_2 \cup X_3), Y_2 \cap Y_3) \\ = \ & (X_1 \cap h^{R^{\max}} f^{R^{\max}}(X_2 \cup X_3), Y_1 \cup (Y_2 \cap Y_3)) \\ = \ & (h^{R^{\max}} f^{R^{\max}}(X_1 \cap (X_2 \cup X_3)), (Y_1 \cup Y_2) \cap (Y_1 \cup Y_3)) \\ = \ & (h^{R^{\max}} f^{R^{\max}}((X_1 \cap X_2) \cup (X_1 \cap X_3)), \\ & (Y_1 \cup Y_2) \cap (Y_1 \cup Y_3)) \\ = \ & (X_1 \cap X_2, Y_1 \cup Y_2) \sqcup (X_1 \cap X_3, Y_1 \cup Y_3) \\ = \ & ((X_1, Y_1) \sqcap (X_2, Y_2)) \sqcup ((X_1, Y_1) \sqcap (X_3, Y_3)). \end{aligned}$$

Hence, $\mathcal{L}(R^{\max})$ is a a distributive lattice.

By using the above propositions, we have the following corollary.

Corollary 3.2. For any concepts (X, Y), (X_1, Y_1), $(X_2, Y_2) \in \mathcal{L}(R^{\max})$, there are

- $\sim\sim (X, Y) = (X, Y)$
- $\sim ((X_1, Y_1) \sqcup (X_2, Y_2))$
  $= \sim (X_1, Y_1) \sqcap \sim (X_2, Y_2)$
- $\sim ((X_1, Y_1) \sqcap (X_2, Y_2))$
  $= \sim (X_1, Y_1) \sqcup \sim (X_2, Y_2)$
- $(X, Y) \sqcup (\sim (X, Y) \sqcap (X_1, Y_1))$
  $= (X, Y) \sqcup (X_1, Y_1)$
- $(X, Y) \sqcap (\sim (X, Y) \sqcup (X_1, Y_1))$
  $= (X, Y) \sqcap (X_1, Y_1)$ and
- $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$
  $\Leftrightarrow \sim (X_2, Y_2) \sqsubseteq \sim (X_1, Y_1)$.

For any two concepts in $R^{\max}$, the union of their intents is an intent of some concept. However, the union of their extents generally does not result in an extent.

Proposition 3.4. For any concepts $(X_1, Y_1), (X_2, Y_2) \in \mathcal{L}(R^{\max})$, if $(X_1, Y_1) \not\sqsubseteq (X_2, Y_2)$ and $(X_2, Y_2) \not\sqsubseteq (X_1, Y_1)$, then $X_1 \cup X_2$ is not an extent of any concept in $R^{\max}$.

Proof: For any $X \subseteq U^{\max}$, we have that $X \subseteq h^{R^{\max}} f^{R^{\max}}(X)$. Because X is an extent of some concept in $R^{\max}$ if and only if $X = h^{R^{\max}} f^{R^{\max}}(X)$, we only need show that $X_1 \cup X_2 \subset h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_2)$. Because

$$\begin{aligned} & (X_1, Y_1) \not\sqsubseteq (X_2, Y_2) \text{ and } (X_2, Y_2) \not\sqsubseteq (X_1, Y_1) \\ \Leftrightarrow \ & X_1 X_2 \text{ and } X_2 X_1 \\ \Leftrightarrow \ & Y_2 Y_1 \text{ and } Y_1 Y_2 \\ \Leftrightarrow \ & Y_1 \cap Y_2 \subset Y_2 \text{ and } Y_1 \cap Y_2 \subset Y_1, \end{aligned}$$

there exist an attribute $a_1 \in Y_1 - Y_1 \cap Y_2$ and an attribute $a_2 \in Y_2 - Y_1 \cap Y_2$. Thereby, we construct a tuple r as follows: for any attribute $a \in A$,

$$r(a) = \begin{cases} 1 & \text{if } a \in Y_1 \cap Y_2, \\ 0 & \text{if } a = a_1, \\ 0 & \text{if } a = a_2, \\ 0 & \text{else.} \end{cases}$$

Obviously, $r \in h^{R^{\max}}(Y_1 \cap Y_2) = h^{R^{\max}} f^{R^{\max}}(X_1 \cup X_2)$. By the construction of r, we have that $r(a_1) = 0$ and $r(a_2) = 0$, and thereby $r \notin X_1$ and $r \notin X_2$, and consequently $r \notin X_1 \cup X_2$.

IV. The connection between concepts in $R^{\max}$ and concepts in R

Firstly, we discuss the structural connection between the concept lattice of a sub-relation and the concept lattice of the maximal relation. Secondly, we provide two algorithms to extract concepts in R from $\mathcal{L}(R^{\max})$.

A. The connection between concept lattice $\mathcal{L}(R^{\max})$ and concept lattice $\mathcal{L}(R)$

Given a relation R based on the schema $S(A)$, for any concept (X, Y) in R, there is a concept $(h^{R^{\max}}(Y), Y)$ in $R^{\max}$. Thus, there exists a map σ_1 from $\mathcal{L}(R)$ to $\mathcal{L}(R^{\max})$, which keeps the intents unchanged.

Proposition 4.1. For any concept $(X, Y) \in \mathcal{L}(R)$,

$$\sigma_1((X, Y)) = (h^{R^{\max}}(Y), Y)$$

is a supremum-preserving order-embedding map.

Proof: Firstly, it is easily inferred that σ_1 is a map from $\mathcal{L}(R)$ to $\mathcal{L}(R^{\max})$. Secondly, σ_1 is order-embedding: for any concepts (X_1, Y_1), $(X_2, Y_2) \in \mathcal{L}(R)$, there are

$$\begin{aligned} & (X_1, Y_1) \sqsubseteq (X_2, Y_2) \\ \Leftrightarrow\ & Y_2 \subseteq Y_1 \\ \Leftrightarrow\ & (h^{R^{\max}}(Y_1), Y_1) \sqsubseteq (h^{R^{\max}}(Y_2), Y_2) \\ \Leftrightarrow\ & \sigma_1((X_1, Y_1)) \sqsubseteq \sigma_1((X_2, Y_2)). \end{aligned}$$

Thirdly, σ_1 is supremum-preserving: for any concepts (X_1, Y_1), $(X_2, Y_2) \in \mathcal{L}(R)$, there are

$$\begin{aligned} & \sigma_1((X_1, Y_1) \sqcup (X_2, Y_2)) \\ =\ & \sigma_1((h^R(Y_1 \cap Y_2), Y_1 \cap Y_2)) \\ =\ & (h^{R^{\max}}(Y_1 \cap Y_2), Y_1 \cap Y_2) \\ =\ & (h^{R^{\max}}(f^{R^{\max}} h^{R^{\max}}(Y_1) \cap f^{R^{\max}} h^{R^{\max}}(Y_2)), Y_1 \cap Y_2) \\ =\ & (h^{R^{\max}} f^{R^{\max}}(h^{R^{\max}}(Y_1) \cup h^{R^{\max}}(Y_2)), Y_1 \cap Y_2) \\ =\ & (h^{R^{\max}}(Y_1), Y_1) \sqcup (h^{R^{\max}}(Y_2), Y_2) \\ =\ & \sigma_1((X_1, Y_1)) \sqcup \sigma_1((X_2, Y_2)). \end{aligned}$$

However, σ_1 is not an infimum-preserving map, as shown in the following example:

Example 4.1. Given the following relation $R = (U, A, I)$, R is a sub-relation of $R^{\max}$ in example 3.1, where $U = \{r_4, r_6, r_7\}$:

$$R = \begin{array}{c|ccc} & a & b & c \\ \hline r_4 & 1 & 0 & 0 \\ r_6 & 0 & 1 & 0 \\ r_7 & 0 & 0 & 1 \end{array}$$

There are concepts $(\{r_4\}, \{a\})$ and $(\{r_6\}, \{b\})$ in R, and the supremum of them is the following concept:

$$(\{r_4\}, \{a\}) \sqcap (\{r_6\}, \{b\}) = (\emptyset, \{a, b, c\}).$$

Because

$$\begin{aligned} & \sigma_1((\{r_4\}, \{a\})) = (\{r_1, r_2, r_3, r_4\}, \{a\}), \\ & \sigma_1((\{r_6\}, \{b\})) = (\{r_1, r_2, r_5, r_6\}, \{b\}), \\ & \sigma_1((\emptyset, \{a, b, c\})) = (\{r_1\}, \{a, b, c\}), \end{aligned}$$

there are

$$\begin{aligned} & \sigma_1((\{r_4\}, \{a\}) \sqcap (\{r_6\}, \{b\})) \\ =\ & \sigma_1((\emptyset, \{a, b, c\})) \\ =\ & (\{r_1\}, \{a, b, c\}) \\ \neq\ & (\{r_1, r_2\}, \{a, b\}) \\ =\ & (\{r_1, r_2, r_3, r_4\}, \{a\}) \sqcap (\{r_1, r_2, r_5, r_6\}, \{b\}) \\ =\ & \sigma_1((\{r_4\}, \{a\})) \sqcap \sigma_1((\{r_6\}, \{b\})). \end{aligned}$$

Conversely, given a concept (X, Y) in $R^{\max}$, for the relation R, $X \cap U$ is the largest set of tuples which have all attributes in Y, and hence $(X \cap U, f^R(X \cap U))$ is a concept in R. Thus, we can construct a map σ_2 from $\mathcal{L}(R^{\max})$ to $\mathcal{L}(R)$, as shown the next proposition:

Proposition 4.2. For any concept $(X, Y) \in \mathcal{L}(R^{\max})$,

$$\sigma_2((X, Y)) = (X \cap U, f^R(X \cap U))$$

is a surjective infimum-preserving order-preserving map from $\mathcal{L}(R^{\max})$ to $\mathcal{L}(R)$.

Proof: Firstly, it is easily inferred that σ_2 is a map from $\mathcal{L}(R^{\max})$ to $\mathcal{L}(R)$. Secondly, σ_2 is order-preserving: for any (X_1, Y_1), $(X_2, Y_2) \in \mathcal{L}(R^{\max})$, we have that

$$\begin{aligned} & (X_1, Y_1) \sqsubseteq (X_2, Y_2) \\ \Leftrightarrow\ & X_1 \subseteq X_2 \\ \Rightarrow\ & X_1 \cap U \subseteq X_2 \cap U \\ \Rightarrow\ & f^R(X_2 \cap U) \subseteq f^R(X_1 \cap U) \\ \Leftrightarrow\ & (X_1 \cap U, f^R(X_1 \cap U)) \sqsubseteq (X_2 \cap U, f^R(X_2 \cap U)) \\ \Leftrightarrow\ & \sigma_2((X_1, Y_1)) \sqsubseteq \sigma_2((X_2, Y_2)). \end{aligned}$$

Thirdly, σ_2 is surjective: for any concept (X, Y) in R, we obtain a concept $(h^{R^{\max}}(Y), Y)$ in $R^{\max}$. It is easily inferred that $h^{R^{\max}}(Y) \cap U = X$, and then

$$\begin{aligned} & \sigma_2((h^{R^{\max}}(Y), Y)) \\ =\ & (R^{\max}(Y) \cap U, f^R(h^{R^{\max}}(Y) \cap U)) \\ =\ & (X, Y). \end{aligned}$$

Fourthly, σ_2 is infimum-preserving: for any concepts (X_1, Y_1), $(X_2, Y_2) \in \mathcal{L}(R^{\max})$, we easily infer that $(X_1 \cap U, f^R(X_1 \cap U))$ and $(X_2 \cap U, f^R(X_2 \cap U))$ are concepts in R, and there are

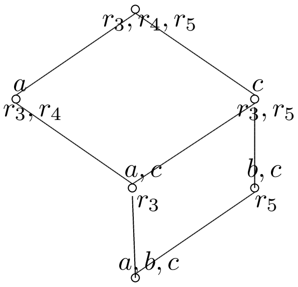
$$\begin{aligned} & \sigma_2((X_1, Y_1) \sqcap (X_2, Y_2)) \\ =\ & \sigma_2((X_1 \cap X_2, Y_1 \cup Y_2)) \\ =\ & (X_1 \cap X_2 \cap U, f^R(X_1 \cap X_2 \cap U)) \\ =\ & ((X_1 \cap U) \cap (X_2 \cap U), f^R(X_1 \cap X_2 \cap U)) \\ =\ & ((X_1 \cap U) \cap (X_2 \cap U), f^R((X_1 \cap U) \cap (X_2 \cap U))) \\ =\ & ((X_1 \cap U) \cap (X_2 \cap U), \\ & f^R(h^R f^R(X_1 \cap U) \cap h^R f^R(X_2 \cap U))) \\ =\ & ((X_1 \cap U) \cap (X_2 \cap U), \\ & f^R(h^R(f^R(X_1 \cap U) \cup f^R(X_2 \cap U)))) \\ =\ & (X_1 \cap U, f^R(X_1 \cap U)) \sqcap (X_2 \cap U, f^R(X_2 \cap U)) \\ =\ & \sigma_2((X_1, Y_1)) \sqcap \sigma_2((X_2, Y_2)). \end{aligned}$$

However, σ_2 is not supremum-preserving, that is, σ_2 is not a homomorphism, see the next example:

Example 4.2. Given the following relation $R = (U, A, I)$, R is a sub-relation of $R^{\max}$ in example 3.1., where $U = \{r_3, r_4, r_5\}$:

$$R = \begin{array}{c|ccc} & a & b & c \\ \hline r_3 & 1 & 0 & 1 \\ r_4 & 1 & 0 & 0 \\ r_5 & 0 & 1 & 1 \end{array}$$

The concept lattice of the relation is as follows:



There are the following concepts in $R^{\max}$:

$$(X_1, Y_1) = (\{r_1, r_2\}, \{a, b\})$$
$$(X_2, Y_2) = (\{r_1, r_3\}, \{a, c\}),$$

and the supremum of them is the following concept

$$(X_1, Y_1) \sqcup (X_2, Y_2) = (\{r_1, r_2, r_3, r_4\}, \{a\}).$$

Because $U = \{r_3, r_4, r_5\}$, there are

$$\sigma_2((X_1, Y_1)) = (\emptyset, \{a, b, c\})$$
$$\sigma_2((X_2, Y_2)) = (\{r_3\}, \{a, c\})$$
$$\sigma_2((X_1, Y_1) \sqcup (X_2, Y_2)) = (\{r_3, r_4\}, \{a\}),$$

and thereby

$$\begin{aligned} & \sigma_2((X_1, Y_1) \sqcup (X_2, Y_2)) \\ = \; & (\{r_3, r_4\}, \{a\}) \\ \neq \; & (\{r_3\}, \{a, c\}) \\ = \; & (\emptyset, \{a, b, c\}) \sqcup (\{r_3\}, \{a, c\}) \\ = \; & \sigma_2((X_1, Y_1)) \sqcup \sigma_2((X_2, Y_2)). \end{aligned}$$

Generally, there is not a surjective homomorphism from $\mathcal{L}(R^{\max})$ to $\mathcal{L}(R)$. For example, for $R^{\max}$ in example 3.1. and R in example 4.1, there does not exist a surjective homomorphism from $\mathcal{L}(R^{\max})$ to $\mathcal{L}(R)$.

B. Extracting concepts in R from concept lattice $\mathcal{L}(R^{\max})$

In this section, we provide two algorithms to extract concepts in R from the concept lattice of $R^{\max}$, which are equivalent. One is based on extents, which means that for any extent X in $R^{\max}$, $X \cap U$ is an extent in R, and the other is based on intents, which means that for any intent Y in $R^{\max}$, if there is no any attribute depending on Y in R, then Y is also an intent in R. By using these two algorithms, one can easily extract all concepts in R.

Algorithm 1 (based on extents)

Input : a relation $R = (U, A, I)$ and the concept lattice $\mathcal{L}(R^{\max})$

Output : all concepts in R

Process :

1. $\Sigma \leftarrow \{\}$
2. *For all concepts* $(X, Y) \in \mathcal{L}(R^{\max})$, *Let* $[(X, Y)]_R = \{(X', Y') \in \mathcal{L}(R^{\max}) | X' \cap U = X \cap U\}$
3. $\Sigma \leftarrow [(X, Y)]_R$
4. *For any* $[(X, Y)]_R \in \Sigma, (X \cap U, Y_R) \in \mathcal{L}(R)$, *where* $Y_R = \bigcup_{(X', Y') \in [(X,Y)]_R} Y'$.

Algorithm 1 is sound, as shown in the following proposition:

Proposition 4.3. Given a relation $R = (U, A, I)$, for any concept $(X, Y) \in \mathcal{L}(R^{\max})$, let $(X_i, Y_i)(i = 1, \ldots, k) \in \mathcal{L}(R^{\max})$ be all concepts such that $X_i \cap U = X \cap U \neq \emptyset$, where k is some natural number. Then $(X \cap U, \bigcup_{i=1}^k Y_i)$ is a concept in R.

Proof: We must show that $h^R(\bigcup_{i=1}^k Y_i) = X \cap U$ and $f^{(R)}(X \cap U) = \bigcup_{i=1}^k Y_i$.

Firstly, $h^R(\bigcup_{i=1}^k Y_i) = X \cap U$. On the one hand, for any tuple $r \in X \cap U$, we have that $r \in \bigcap_{i=1}^k X_i$. For any attribute $a \in \bigcup_{i=1}^k Y_i$, there exists some Y_j such that $a \in Y_j$. Because (X_j, Y_j) is a concept in $R^{\max}$, $r(a) = 1$ holds in R. Hence, $r \in h^R(\bigcup_{i=1}^k Y_i)$, and thereby $X \cap U \subseteq h^R(\bigcup_{i=1}^k Y_i)$. On the other hand, for any tuple $r \in h^R(\bigcup_{i=1}^k Y_i) \subseteq U$, then for any attribute $a \in \bigcup_{i=1}^k Y_i$, there is $I(r, a) = I^{\max}(r, a) = 1$, i.e., $r(a) = 1$, and hence $r \in \bigcap_{i=1}^k X_i \cap U = X \cap U$. Therefore, $h^R(\bigcup_{i=1}^k Y_i) \subseteq X \cap U$.

Secondly, $f^{(R)}(X \cap U) = \bigcup_{i=1}^k Y_i$. For any $a \in \bigcup_{i=1}^k Y_i$, there exists some Y_j such that $a \in Y_j$. For any tuple $r \in X \cap U = X_j \cap U$, there is $r \in X_j$. Because (X_j, Y_j) is a concept in $R^{\max}$, $r(a) = 1$ holds in R, and hence $\bigcup_{i=1}^k Y_i \subseteq f^{(R)}(X \cap U)$. Conversely, in order to show that $f^{(R)}(X \cap U) \subseteq \bigcup_{i=1}^k Y_i$, we assume that there exists an attribute $a \in f^{(R)}(X \cap U)$ but $a \notin \bigcup_{i=1}^k Y_i$, and hence $(\bigcup_{i=1}^k Y_i) \cup \{a\}$ is an intent of some concept in $R^{\max}$. For any tuple $r \in U$, if r has all attributes in $(\bigcup_{i=1}^k Y_i) \cup \{a\}$, then r has all attributes in $\bigcup_{i=1}^k Y_i$, and hence $r \in \bigcap_{i=1}^k X_i$, and further $r \in (\bigcap_{i=1}^k X_i) \cap U = X \cap U$. It is easily inferred that $X \cap U$ is the largest set of tuples having all attributes in $(\bigcup_{i=1}^k Y_i) \cup \{a\}$. Let

$$h^{R^{\max}}((\bigcup_{i=1}^k Y_i) \cup \{a\}) = \overline{X} = (X \cap U) \cup X_0,$$

where $X_0 \bigcap U = \emptyset$, and X_0 is the set of tuples having all attributes in $(\bigcup_{i=1}^k Y_i) \cup \{a\}$. Hence, $X \cap U = \overline{X} \cap U$, and further $\bigcap_{i=1}^k X_i \subseteq \overline{X}$. However, by using $\bigcup_{i=1}^k Y_i \subset (\bigcup_{i=1}^k Y_i) \cup \{a\}$, we have that $\overline{X} \subset \bigcap_{i=1}^k X_i$, which leads to a contradiction.

$Algorithm\ 2\ (based\ on\ intents)$
$Input: a\ relation\ R = (U, A, I)\ and$ $the\ concept\ lattice\ \mathcal{L}(R^{\max})$
$Output: all\ concepts\ in\ R$
$Process:$ $For\ all\ concepts\ (X, Y) \in \mathcal{L}(R^{\max}),$ $If\ for\ any\ attribute\ a \in A - Y,$ $R \not\models Y \Rightarrow a,$ $Then\ (X \cap U, Y)\ is\ a\ concept\ in\ R.$

Algorithm 2 is sound, as shown in the following proposition:

Proposition 4.4. For any concept $(X, Y) \in \mathcal{L}(R^{\max})$, $(X \cap U,\ Y)$ is a concept in R if and only if for any attribute $a \notin Y$, there is $R \not\models Y \Rightarrow a$.

Proof: ($\Rightarrow$) Because $(X \cap U,\ Y)$ is a concept in R, $Y = f^R h^R(Y)$. Assume that there exists an attribute $a \notin Y$ such that $R \models Y \Rightarrow a$, where $R \models Y \Rightarrow a$ means that for any tuple in U, if r has all attributes in Y, then r also has the attribute a. Because

$$\begin{aligned} & R \models Y \Rightarrow a \\ \Leftrightarrow\ & \forall r(\forall a' \in Y(r(a') = 1) \rightarrow r(a) = 1) \\ \Leftrightarrow\ & \forall r \in h^R(Y)(r(a) = 1) \\ \Leftrightarrow\ & a \in f^R h^R(Y), \end{aligned}$$

$a \in f^R h^R(Y) = Y$, which leads to a contradiction.

($\Leftarrow$) Assume that $(X \cap U, Y)$ is not a concept in R. Because $X \cap U$ is an extent of some concept in R, we have that $f^R(X \cap U) \neq Y$, and further $Y \subset f^R(X \cap U) = f^R h^R(Y)$. Hence, there exists an attribute $a \in f^R h^R(Y)$ but $a \notin Y$, and we have that

$$\begin{aligned} & a \in f^R h^R(Y) \\ \Leftrightarrow\ & \forall r \in h^R(Y)(r(a) = 1) \\ \Leftrightarrow\ & \forall r(\forall a' \in Y(r(a') = 1) \rightarrow r(a) = 1) \\ \Leftrightarrow\ & R \models Y \Rightarrow a, \end{aligned}$$

which is not consist to $R \not\models Y \Rightarrow a$.

Actually, Algorithm 1 and Algorithm 2 are equivalent.

Proposition 4.5. Given a relation $R = (U, A, I)$, for any concepts $(X_1, Y_1), (X_2, Y_2) \in \mathcal{L}(R^{\max})$, if $X_1 \subset X_2$ then $X_1 \cap U = X_2 \cap U$ if and only if for any attribute $a \in Y_1 - Y_2$, $R \models Y_2 \Rightarrow a$ holds.

Proof: ($\Rightarrow$) Assume that there exists an attribute $a \in Y_1 - Y_2$ such that $R \not\models Y_2 \Rightarrow a$. Then there exists a tuple $r \in U$, which has all attributes in Y_2 but $r(a) = 0$. On the one hand, $r \in X_2$ follows directly from $(X_2, Y_2) \in \mathcal{L}(R^{\max})$, and further $r \in X_2 \cap U = X_1 \cap U$, and hence $r \in X_1$. On the other hand, (X_1, Y_1) is a concept in $R^{\max}$, so $r(a) = 1$. This leads to a contradiction. Hence, for any attribute $a \in Y_1 - Y_2$, $R \models Y \Rightarrow a$.

($\Leftarrow$) Assume that $X_1 \cap U \subset X_2 \cap U$. Then there exists a tuple $r \in X_2 \cap U$ but $r \notin X_1 \cap U$, and hence $r \notin X_1$. Because $r \in X_2 \cap U$ and (X_2, Y_2) is a concept, r has all attributes in Y_2, and further r has all attributes in Y_1, and hence $r \in X_1$, which leads to a contradiction.

V. Conclusion

In this paper, we firstly analyzed concepts in $R^{\max}$ and the concept lattice $\mathcal{L}(R^{\max})$, and obtained some important results, which generally do not hold for other relations. Secondly, for any relation R based on the schema $S(A)$, we discussed the structural connection between $\mathcal{L}(R^{\max})$ and $\mathcal{L}(R)$. Thirdly, we provided two equivalent algorithm to extract concepts in R from $\mathcal{L}(R^{\max})$. Actually, our methods can be applied to analyze non-binary relations.

Several problems remain to be investigated. Because real world applications often include imprecise and uncertain information, one of the interesting problems is how to capture information on uncertainty and imprecision along with precise values in databases. The future works will focus on these questions and connections among concept lattices of relations, and our methods can be used to some applications such as model and classification [22, 24].

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