An Effective Algorithm for Globally Solving a Class of Linear Fractional Programming Problem

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***Abstract*—This article presents a branch-and-bound algorithm for globally solving the general linear sum of ratios problem (GFP). By utilizing equivalent transformation and linearization technique, a linear relaxation programming (LRP) of original problem is constructed. The algorithm economizes the required computations by conducting the branch-and-bound search in R^p, rather than in R^n, where p is the number of ratios in the objective function of problem (P) and n is the number of decision variables in problem (P). To implement the algorithm, the main computations involve solving a sequence of linear programming problems for which simplex algorithm are available. Numerical experiments are given to demonstrate that the proposed algorithm can systematically solve problem (GFP) to find the global optimum.**

***Index Terms*—global optimization, linear relaxation, branch and bound, fractional programming, sum-of-ratios**

I. Introduction

Consider the general linear sum of ratios problem

$$(\text{GFP}):\begin{cases} v=\max g(x)=\displaystyle\sum_{i=1}^{p}\frac{\sum_{j=1}^{n}c_{ij}x_j+d_i}{\sum_{j=1}^{n}e_{ij}x_j+f_i} \\ s.t.\quad Ax\le b, x\ge 0 \end{cases}$$

where $A\in R^{m\times n}, b\in R^m, c_{ij}, d_i, e_{ij}, f_i$ are all arbitrary real number,

$$\Lambda \triangleq \{x\in R^n \mid Ax\le b, x\ge 0\}$$

is bounded with $\text{int}\,\Lambda\ne\phi$, and for $\forall x\in\Lambda,\ i=1,\ldots,p,\ j=1,\ldots,n,$ we have

$$\sum_{j=1}^{n}e_{ij}x_j+f_i\ne 0.$$

Problem (GFP) has attracted the interest of practitioners and researchers for at least 40 years. This is because, from a practical point of view, problem (GFP) and special cases of problem (GFP) have a number of important applications. Included among these are multistage shipping problems (Ref. [1]), certain government contracting problems (Ref. [2]), and various economic and financial problems (Refs. [3-7, 22, 23]). In these problems, the number of terms p in the objective function of problem (GFP) is usually less than the number of the number of decision variables terms n. From a research point of view, problem (GFP) poses significant theoretical and computational difficulties. This is mainly due to the fact that problem (GFP) is a global optimization problem, i.e., it is known to generally possess multiple local optima that are not globally optimal [24, 25].

Many algorithms have been developed for solving the special case of general linear sums of ratios problems (GFP), which are intended only for the sum of linear ratios problem with the assumption that

$$\sum_{j=1}^{n}c_{ij}x_j+d_i\ge 0,$$

and

$$\sum_{j=1}^{n}e_{ij}x_j+f_i>0 \text{ for any } x\in\Lambda.$$

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In this case, for instance, if the number of ratios $p=2$, the simplex method-based global solution algorithms by Cambini et al. (Ref. [8]) and Konno et al (Ref. [9]) can be used. When the number of ratios $p \ge 2$, the global solution algorithms by Falk and Palocsay (Ref. [10]), Konno and Yamashita (Ref. [11]), Konno and Fukaishi (Ref. [12]), and Kuno (Ref. [13]), H. Konno and H. Yamashita (Ref. [2]), J.E. Falk and S.W. Palocsay (Ref. [3]) are available. If exactly the number of ratios $p=3$, the heuristic algorithm of Konno and Abe (Ref.[14]) may be employed. To solve sums of ratios problems in which the numerators and denominators are affine functions and the feasible region is a compact convex set, an algorithm of Konno et al (Ref. [15]) can be used. In addition, under the assumption that

$$\sum_{j=1}^{n} c_{ij} x_j + d_i \ge 0 ,$$

and

$$\sum_{j=1}^{n} e_{ij} x_j + f_i \ne 0 ,$$

a branch and bound algorithm has been proposed (Ref. [16], Refs. [17-19]).

Recently, H. Benson (Ref. [20]) consider the sum-of-ratios fractional program

$$\theta = \max \sum_{i=1}^{p} \frac{<n_i, y> + g_i}{<d_i, y> + h_i}, \quad s.t.\ y \in Y , \qquad (1)$$

where $p \ge 2$, Y is a nonempty, compact convex set in R^n, and for each $i = 1,2,\cdots,p$, $n_i, d_i \in R^n$, $g_i, h_i \in R$, and $\left(<n_i, y> + g_i\right)$ and $\left(<d_i, y> + h_i\right)$ are positive for all $y \in Y$. Notice that under these assumptions, the global maximum θ for problem (1) is attained at one or more points in Y.

In Ref. [20], the author presented a branch and bound-outer approximation algorithm for globally solving the above problem (1). To globally solve problem (1), the algorithm instead globally solves an equivalent problem that seeks to minimize an indefinite quadratic function over a nonempty, compact convex set. To solve this problem, the algorithm combines a branch and bound search with an outer approximation method. From a computational point of view, the main work of the algorithm involves solving a sequence of lower bounding convex relaxation programming problems. Since the feasible regions of these convex programs are identical to one another except for certain linear constraints, to solve them, an optimal solution to one problem can potentially be used as an effective starting solution for the next problem.

For each $i = 1,2,\cdots,p$, Benson [20] let

$$U_0 = \max \frac{\sqrt{<n_i, y> + g_i}}{<d_i, y> + h_i}, \quad s.t.\ y \in Y . \qquad (2)$$

From (Ref. [17]), for each $i = 1,2,\cdots,p$, the optimal value U_i^0 in (2) is positive and is always attained, and this value can be computed by applying any efficient convex programming algorithm to the optimization problem in (2); i.e., any local maximum of problem (2) is a global maximum. In addition, for each $i = 1,2,\cdots,p$, let

$$S_i = \max[<d_i, y> + h_i], \quad s.t.\ y \in Y . \qquad (3)$$

and

$$T_i = \max \sqrt{<n_i, y> + g_i}, \quad s.t.\ y \in Y . \qquad (4)$$

Then, for each $i = 1,2,\cdots,p$, it is evident that the value of S_i in (3) can be computed by solving a convex programming problem. It is not difficult to show that for each $i = 1,2,\cdots,p$, the value of T_i in (4) is also given by solving a convex program.

Let

$$H^0 = \left\{ \begin{matrix} (u,v) \in R^{2p} \mid 0 \le u_i \le U^0{}_i, \\ 0 \le v_i \le \left(U^0{}_i\right)^2, i = 1,2,\cdots,p \end{matrix} \right\}$$

and let

$$Z = \left\{ (t,s,y) \in R^{2p+n} \mid y \in Y \text{ and } (5)-(8) \text{ hold} \right\}$$

with (5)–(8) given by

$$t_i + \sqrt{[<n_i, y> + g_i]} \ge 0, i = 1,2,\cdots,p \qquad (5)$$

$$s_i - <d_i, y> + h_i, i = 1,2,\cdots,p \qquad (6)$$

$$-T_i \le t_i \le 0, i = 1,2,\cdots,p \qquad (7)$$

$$0 \le s_i \le S_i, i = 1,2,\cdots,p \qquad (8)$$

Benson [20] consider the problem (K) given by

$$(K) \begin{cases} \gamma = \min \sum_{i=1}^{p} 2u_i t_i + v_i s_i, \\ s.t.\ -u_i^2 + v_i \ge 0, i = 1,2,\cdots,p \\ \quad (t,s,y) \in Z, \\ \quad (u,v) \in H^0. \end{cases} \qquad (9)$$

By the conclusion of theorem 2.1, we know that problem (1) and problem (K) is equivalent problem, Then the main work of Ref. [20] will solve problem (K).

In this article, we present an effective branch-and-bound algorithm for globally solving a general linear sum of ratios problem (GFP) by solving a sequence of linear programming problem over partitioned subsets. The main feature of this algorithm is as follows. Firstly in (GFP), we only request

$$\sum_{j=1}^{n} e_{ij} x_j + f_i \ne 0 ,$$

then the model of this paper is more general than other paper considered. Secondly, the algorithm economizes the required computations by conducting the branch-and-bound search in the space R^p, rather than in the space R^n or R^{2p}. Thirdly, to implement the algorithm, the main computations involve solving a sequence of linear programming problems which is easy to be obtained than in [5] and does not generate new variables, for which standard algorithms are available. Fourthly, the proposed branch and bound algorithm is convergent to the global maximum through the successive refinement of the linear relaxation of feasible region of the objection function and constraint functions and the solutions of a series of LRP. At last, numerical experiments are given to show the feasibility of our algorithm.

The article is organized as follows. In Section 2, by using a transformation technique, problem EP is derived that is equivalent to problem GFP. The rectangular branching process, the upper and lower bounding process used in this approach are defined and studied in Section 3. The algorithm is introduced in Section 4, and its convergence is shown. Section 5 report some numerical results obtained by solving some examples. Finally, the summary of this paper is given.

II. PRELIMINAIRES

In this section, we first give an important theorem, which is the foundation of the global optimization algorithm.

Theorem 1. Assume $\sum_{j=1}^{n} e_{ij}x_j + f_i \neq 0$ for $\forall x \in \Lambda$, then

$$\sum_{j=1}^{n} e_{ij}x_j + f_i > 0$$

or

$$\sum_{j=1}^{n} e_{ij}x_j + f_i < 0 .$$

Proof. By the intermediate value theorem, the conclusion is obvious.

For $\forall x \in \Lambda$, if

$$\sum_{j=1}^{n} e_{ij}x_j + f_i < 0 ,$$

then we have

$$\frac{\sum_{j=1}^{n} c_{ij}x_j + d_i}{\sum_{j=1}^{n} e_{ij}x_j + f_i} = \frac{-\left(\sum_{j=1}^{n} c_{ij}x_j + d_i\right)}{-\left(\sum_{j=1}^{n} e_{ij}x_j + f_i\right)} \quad (1)$$

Obviously, in (1), denominators are all positive. Hence, in problem GFP, we can assume

$$\sum_{j=1}^{n} e_{ij}x_j + f_i > 0$$

is always holds. In addition, since

$$\frac{\sum_{j=1}^{n} c_{ij}x_j + d_i}{\sum_{j=1}^{n} e_{ij}x_j + f_i} = \frac{\sum_{j=1}^{n} c_{ij}x_j + d_i + M_i\left(\sum_{j=1}^{n} e_{ij}x_j + f_i\right)}{\sum_{j=1}^{n} e_{ij}x_j + f_i} = M_i$$

where $M_i (i = 1,\ldots, p)$ is a positive number, if $M_i (i = 1,\ldots, p)$ large enough,

$$\sum_{j=1}^{n} c_{ij}x_j + d_i + M_i\left(\sum_{j=1}^{n} e_{ij}x_j + f_i\right) > 0$$

can be satisfied. Therefore, in the following, without loss of generality, we can assume that

$$\sum_{j=1}^{n} c_{ij}x_j + d_i \geq 0$$

and

$$\sum_{j=1}^{n} e_{ij}x_j + f_i > 0$$

in the GFP.

Next, we show how to convert problem GFP into an equivalent problem EP.

For each $i = 1,\ldots, p,$ let

$$l_i = \min_{x\in\Lambda} \sum_{j=1}^{n} e_{ij}x_j + f_i ,$$

$$u_i = \max_{x\in\Lambda} \sum_{j=1}^{n} e_{ij}x_j + f_i$$

Define

$$H_0 = \left\{ y \in R^p \mid l_i^0 \leq y_i \leq u_i^0, i = 1,\ldots p \right\} ,$$

then problem GFP can be converted into an equivalent nonconvex programming problem as follows:

$$\mathrm{EP}(H_0): \begin{cases} v(H_0) = \max \varphi_0(x, y) \\ \qquad = \sum_{i=1}^{p} \frac{\sum_{j=1}^{n} c_{ij}x_j + d_i}{y_i} \\ s.t.\ \varphi_i(x, y) = \sum_{j=1}^{n} e_{ij}x_j + f_i - y_i \leq 0, \quad i = 1,\ldots p, \\ \quad x \in \Lambda, y \in H_0. \end{cases}$$

The key equivalence result for problem GFP and $\mathrm{EP}(H_0)$ is given by the following theorem.

Theorem 2. If $(x^*, y_1^*,\ldots, y_p^*)$ is a global optimal solution for problem $\mathrm{EP}(H_0)$, then x^* is a global optimal solution for problem GFP. Converse, if x^* is a global optimal solution for problem GFP, then $(x^*, y_1^*,\ldots, y_p^*)$ is a global optimal solution for problem $\mathrm{EP}(H_0)$, where

$$y_i^* = \sum_{j=1}^{n} e_{ij}x_j + f_i, i = 1,\ldots, p .$$

Proof. The proof of this theorem follows easily from the definitions of problems GFP and $\mathrm{EP}(H_0)$, therefore, it is omitted.

From Theorem 2, in order to globally solve problem GFP, we may globally solving problem $\mathrm{EP}(H_0)$ instead.

III. BASIC OPERATIONS

In this section, based on the above equivalent problem, a branch and bound algorithm is proposed for solving the global optimal solution of GFP. The main idea of this algorithm consists of three basic operations: successively refined partitioning of the feasible set, estimation of upper and lower bounds for the optimal value of the objective function. Next, we begin the establishment of algorithm with the basic operations needed in a branch and bound scheme.

A. Branching Process

In this algorithm, the branching process is performed in R^p, rather in R^n, that iteratively subdivides the p-dimensional rectangle H_0 of problem EP(H_0) into smaller subrectangles that are also of dimension p. Let

$$H = \left\{ y \in R^p \mid l_i \le y_i \le u_i, i = 1, \dots p \right\}$$

denote the initial rectangle H_0 or subrectangle of it, the branching rule as follows:

(i) Let $\tau_i = \frac{1}{2}(l_i + u_i), i = 1, \dots, p$,

(ii) Let

$$H^1 = \left\{ y \in R^p \mid l_i \le y_i \le \tau_i, i = 1, \dots p \right\}$$

$$H^2 = \left\{ y \in R^p \mid \tau_i \le y_i \le u_i, i = 1, \dots p \right\}$$

It follows easily that this branching process is exhaustive, i.e. if $\left\{ H^k \right\}$ denotes a nested subsequence of rectangles (i.e. $H^{k+1} \subseteq H^k$ for all k) formed by branching process, then there exists a unique point $y \in R^p$ such that

$$\bigcap_k H^k = \left\{ y \right\}.$$

B. Upper Bound and Lower Bound

For each rectangle

$$H = \left\{ y \in R^p \mid l_i \le y_i \le u_i, i = 1, \dots p \right\} \qquad (H \subseteq H_0)$$

formed by the branching process, the upper bound process is used to compute an upper bound UB(H) for the optimal value $v(H)$ of problem EP(H).

$$\mathrm{EP}(H): \begin{cases} v(H) = \max \varphi_0(x, y) = \sum_{i=1}^{p} \frac{\sum_{j=1}^{n} c_{ij} x_j + d_i}{y_i} \\ s.t.\ \ \varphi_i(x, y) = \sum_{j=1}^{n} e_{ij} x_j + f_i - y_i \le 0, i = 1, \dots p, \\ \quad x \in \Lambda, y \in H. \end{cases}$$

It will be seen from below, the upper bound UB(H) can be found by solving an ordinary linear program.

In the following, for convenience of expression, let

$$T_i^+ = \left\{ j \mid c_{ij} > 0, j = 1, \dots, n \right\}, j = 1, \dots, p$$

$$T_i^- = \left\{ j \mid c_{ij} < 0, j = 1, \dots, n \right\}, j = 1, \dots, p$$

$$D^+ = \left\{ j \mid d_i > 0, i = 1, \dots, n \right\},$$

$$D^- = \left\{ j \mid d_i < 0, i = 1, \dots, n \right\},$$

$$E_i^+ = \left\{ j \mid e_{ij} > 0, j = 1, \dots, n \right\}, j = 1, \dots, p,$$

$$E_i^- = \left\{ j \mid e_{ij} < 0, j = 1, \dots, n \right\}, j = 1, \dots, p,$$

First, consider objective function $\varphi_0(x, y)$, we have

$$\begin{aligned} \varphi_0(x, y) &= \sum_{i=1}^{p} \frac{\sum_{j=1}^{n} c_{ij} x_j + d_i}{y_i} \\ &\le \sum_{j=1}^{n} \left(\sum_{j \in T_i^+} c_{ij} \frac{x_j}{l_i} + \sum_{j \in T_i^-} c_{ij} \frac{x_j}{u_i} \right) + \sum_{i \in D^+} \frac{d_i}{l_i} + \sum_{i \in D^-} \frac{d_i}{u_i} \\ &\triangleq \varphi_0^u(x) \end{aligned}$$

Then, consider constraint function $\varphi_i(x, y), i = 1, \dots, p$,

$$\begin{aligned} \varphi_i(x, y) &= \sum_{j=1}^{n} e_{ij} x_j + f_i - y_i \\ &\le \sum_{j=1}^{n} e_{ij} x_j + f_i - u_i \\ &\triangleq \varphi_i^l(x), \end{aligned}$$

Based on the above discussion, we can construct a linear relaxation programming (LRP) as follows, which provides an upper bound for the optimal value $v(H)$ of problem EP(H).

$$\mathrm{LRP}(H): \begin{cases} UB(H) = \max \varphi_0^u(x, y) \\ s.t.\ \ \varphi_i^l(x, y) \le 0, i = 1, \dots p, \\ \quad x \in \Lambda. \end{cases}$$

Remark 1. Let $v[p]$ denotes the optimal value of the problem p, then, from the above discussion, the optimal values of LRP(H) and EP(H) satisfy

$$v[LRP(H)] \ge v[EP(H)] \text{ for } \forall H \in H_0.$$

Remark 2. Obviously, if

$$\overline{H} \subseteq H \subseteq H_0,$$

then

$$UB\left(\overline{H}\right) \le UB\left(H\right).$$

Another basic operation is to determinate a lower bound for the optimal value $v(H_0)$ of problem EP(H_0). By the upper bound process, through solving LRP(H), we will have a optimal solution x^*.

Let

$$y^* = \sum_{j=1}^{n} e_{ij} x_j^* + f_i,$$

obviously, $\left(x^*, y^* \right)$ is a feasible solution of problem EP(H_0), hence, $\varphi_0(x^*, y^*)$ provides a lower bound for the optimal value $v(H_0)$ of problem EP(H_0).

IV. Algorithm and its Convergence

Based upon the results and operations given in Section 3, the branch and bound algorithm for problem GFP may be stated as follows.

Branch and bound algorithm

Step 0. Choose $\varepsilon \ge 0$. Let H_0 be denoted by

$$H_0 = \left\{ y \in R^p \mid l_i^0 \le y_i \le u_i^0, i = 1, \dots p \right\},$$

Find an optimal solution x^0 and the optimal value UB(H_0) for problem LRP(H_0). Set

$$\mathrm{UB}_0 = \mathrm{UB}(H^0),$$

$$x^c = x^0 .$$

Set

$$y_i^c = \sum_{j=1}^{n} e_{ij} x_j + f_j, i = 1, 2, \ldots, p,$$

$$LB_0 = \varphi_0(x^c, y^c),$$

If

$$UB_0 - LB_0 \le \varepsilon ,$$

stop. (x^c, y^c) and x^c are global ε-optimal solutions for problems $\mathrm{EP}(H_0)$ and GFP, respectively. Otherwise, set

$$P_0 = \{H_0\} , \ F = \phi, \ k = 1 ,$$

and go to Step 1.

Step 1. Set

$$LB_k = LB_{k-1} .$$

Subdivide H^{k-1} into two p-dimensional rectangles

$$H^{k,1}, H^{k,2} \subseteq R^p ,$$

via the branching rule. Set

$$F = F \bigcup \{H^{k-1}\} .$$

Step 2. For $j = 1, 2$ compute $\mathrm{UB}(H^{k,j})$ and, if

$$\mathrm{UB}(H^{k,j}) \ne -\infty ,$$

find an optimal solution $x^{k,j}$ for problem $\mathrm{LRP}(\hat{H})$ with $(\hat{H}) = H^{k,j}$, Set $t = 0$.

Step 3. Set $t = t + 1$. If $t > 2$, go to Step 5. Otherwise, continue.

Step 4. If

$$\mathrm{UB}(H^{k,t}) \le LB_k ,$$

set

$$F = F \bigcup \{H^{k,t}\} ,$$

and go to step 3. Otherwise, Set

$$y_i^{k,t} = \sum_{j=1}^{n} e_{ij} x_j^{k,t} + f_i, i = 1, 2, \ldots, p.$$

Let

$$LB_k = \max \{LB_k, \varphi_0(x^{k,t}, y^{k,t})\}$$

If

$$LB_k > \varphi_0(x^{k,t}, y^{k,t}) ,$$

go to Step 3. If

$$LB_k = \varphi_0(x^{k,t}, y^{k,t})$$

set

$$x^c = x^{k,t}, (x^c, y^c) = (x^{k,t}, y^{k,t})$$

and set

$$F = F \bigcup \{H \in (P_{k-1} \mid UB(H) \le LB_k)\}$$

and continue.

Step 5. Set

$$P_k = \{H \mid H \in (P_{k-1} \bigcup \{H^{k,1}, H^{k,2}\}), H \notin F\}$$

Step 6. Set

$$UB_k = \max\{UB(H) \mid H \in P_k \} ,$$

and let $H^k \in P_k$ satisfy

$$UB_k = UB(H^k) .$$

If

$$UB_k - LB_k \le \varepsilon ,$$

stop. (x^c, y^c) and x^c are global ε-optimal solutions for problems $\mathrm{EP}(H_0)$ and GFP, respectively. Otherwise, set $k = k + 1$ and go to Step 1.

The convergence properties of the algorithm are given in the following theorem.

Theorem 3. (a) If the algorithm is finite, then upon termination, (x^c, y^c) and x^c are global ε-optimal solutions for problems $\mathrm{EP}(H_0)$ and GFP, respectively.

(b) For each $k \ge 0$, let x^k denote the incumbent solution x^c for problem GFP at the end of Step k. If the algorithm is infinite, every accumulation point of which is a global optimal solution for problem GFP, and

$$\lim_{k\to\infty} UB_k = \lim_{k\to\infty} LB_k = v$$

Proof. (a) If the algorithm is finite, then it terminates in Step $k \ge 0$. Upon termination, since (x^c, y^c) is found by solving problem $\mathrm{EP}(H)$, for some $H \subseteq H^0$, for an optimal solution x^c and setting

$$y_i^c = \sum_{j=1}^{n} e_{ij} x_j^c + f_i, i = 1, 2, \ldots, p,$$

x^c is a feasible solution for problem GFP, and (x^c, y^c) is a feasible solution for problem $\mathrm{EP}(H_0)$. Upon termination of the algorithm,

$$UB_k - LB_k \le \varepsilon$$

is satisfied. From Step 0 and Step 1 and Step 4, this implies that

$$UB_k - \varphi_0(x^c, y^c) \le \varepsilon .$$

By the algorithm, it shows that

$$UB_k \ge v$$

Since (x^c, y^c) is a feasible solution for problem $\mathrm{EP}(H_0)$,

$$\varphi_0(x^c, y^c) \le v$$

Taken together, this implies that

$$v \le UB_k \le \varphi_0(x^c, y^c) + \varepsilon \le v + \varepsilon$$

Therefore

$$v - \varepsilon \le \varphi_0(x^c, y^c) \le v .$$

Since

$$y_i^c = \sum_{j=1}^{n} e_{ij} x_j^c + f_i, i = 1, 2, \ldots, p,$$

we have

$$g(x^c) = \varphi_0(x^c, y^c).$$

this implies that

$$v - \varepsilon \le g(x^c) \le v ,$$

and the proof of part (a) is complete.

(b) Suppose that the algorithm is infinite. Then it generates a sequence of incumbent solutions for problem $\mathrm{EP}(H_0)$, which we may denote by $\{(x^k, y^k)\}$. For each $k \ge 1$, $\{(x^k, y^k)\}$ is found by solving problem $\mathrm{EP}(H^k)$, for some rectangle $H^k \subseteq H_0$, for an optimal solution $H^k \in \Lambda$, and setting

$$y_i^k = \sum_{j=1}^{n} e_{ij} x_j^k + f_i, i = 1, 2, \ldots, p, .$$

Therefore, the sequence x^k consists of feasible solutions for problem GFP. Let $\bar{x}$ be an accumulation point of $\{x^k\}$, and assume without loss of generality that

$$\lim_{k\to\infty} x^k = \bar{x} .$$

Then, since Λ is a compact set, $\bar{x} \in \Lambda$. Furthermore, since $\{x^k\}$ is infinite, we may assume that without loss of generality that, for each k, $H^{k+1} \subseteq H^k$. From Horst and Tuy (Ref.[21]), since the rectangles $H^k, k \ge 1$, are formed by rectangular bisection, this implies that, for some point $\bar{y} \in R^p$

$$\lim_{k\to\infty} H^k = \bigcap_k H^k = \{\bar{y}\} .$$

Let $\bar{H} = \{\bar{y}\}$ and, for each k, by Remark 2 and Step 4, this implies that $\left\{\mathrm{UB}(H^k)\right\}$ is a nonincreasing sequence of real numbers bounded below by v. Therefore, $\lim_{k\to\infty} UB(H^k)$ is a finite number and satisfies

$$\lim_{k\to\infty} UB(H^k) \ge v.$$

For each k, from Step 2, $UB(H^k)$ equal to the optimal value of the problem $\mathrm{LRP}(H^k)$ and x^k is an optimal solution for this problem. From the above, we have

$$\lim_{k\to\infty} l^k = \lim_{k\to\infty} u^k = \{\bar{y}\} = \bar{H}.$$

Since

$$\lim_{k\to\infty} x^k = \bar{x} ,$$

$$l_i^k \le \sum_{j=1}^{n} e_{ij} x_j^k + f_i \le u_i^k ,$$

and the continuity of $\sum_{j=1}^{n} e_{ij} x_j^k + f_i$,

$$\sum_{j=1}^{n} e_{ij} \bar{x}_j + f_i = \bar{y}_i, i = 1, 2, \ldots, p .$$

This implies that $(\bar{x}, \bar{y})$ is a feasible solution for problem $\mathrm{EP}(H_0)$. Therefore,

$$\varphi_0(\bar{x}, \bar{y}) \le v .$$

Combing the former formulation, we obtain that

$$\varphi_0(\bar{x}, \bar{y}) \le v \le \lim_{k\to\infty} UB(H^k) .$$

Since

$$\begin{aligned} &\lim_{k\to\infty} UB(H^k) \\ &= \sum_{j=1}^{n} \left(\sum_{j\in T_i^+} c_{ij} \frac{x_j}{l_i^k} + \sum_{j\in T_i^-} c_{ij} \frac{x_j}{u_i^k} \right) + \sum_{i\in D^+} \frac{d_i}{l_i^k} + \sum_{i\in D^-} \frac{d_i}{u_i^k} \\ &= \sum_{i=1}^{p} c_i \frac{\sum_{j=1}^{n} e_{ij} \bar{x}_j + f_j}{\bar{y}_i} \\ &= \varphi_0(\bar{x}, \bar{y}) \end{aligned}$$

From the above formulation, we have

$$\lim_{k\to\infty} UB(H^k) = v = \varphi_0(\bar{x}, \bar{y}) .$$

Therefore, $(\bar{x}, \bar{y})$ is a global optimal solution for problem $\mathrm{EP}(H_0)$. By Theorem 2, this implies that $\bar{x}$ is a global optimal solution for problem GFP.

For each k, since x^k is the incumbent solution for problem GFP at the end of Step k,

$$LB_k = g(x^k) = g(\bar{x}), \text{ for all } k \ge 1 .$$

By the continuity of g, we have

$$\lim_{k\to\infty} g(x^k) = g(\bar{x}).$$

Since $\bar{x}$ is a global optimal solution for problem GFP,

$$g(\bar{x}) = v.$$

Therefore,

$$\lim_{k\to\infty} LB_k = v ,$$

and the proof is complete.

V. Numerical Experiments

To verify the performance of the proposed global optimization algorithm, some test problems are implemented on microcomputer, and the convergence tolerance set to $\varepsilon = 10^{-6}$ in our experiment. The results are summarized in Table 1.

Example 1.

$$\begin{aligned} \min \quad & -\frac{3x_1 + 5x_2 + 3x_3 + 50}{3x_1 + 4x_2 + 5x_3 + 50} - \frac{3x_1 + 4x_2 + 50}{4x_1 + 3x_2 + 2x_3 + 50} - \frac{4x_1 + 2x_2 + 4x_3 + 50}{5x_1 + 4x_2 + 3x_3 + 50} \\ s.t. \quad & 10x_1 + 3x_2 + 8x_3 \le 10, \\ & 6x_1 + 3x_2 + 3x_3 \le 10, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Obtain the optimal solution $x_1^* = 0,$ $x_2^* = 0.333333,$ $x_3^* = 0$.

Example 2.

$$\begin{aligned} \min \quad & -\frac{4x_1 + 3x_2 + 3x_3 + 50}{3x_2 + 3x_3 + 50} - \frac{3x_1 + 4x_3 + 50}{4x_1 + 4x_2 + 5x_3 + 50} - \frac{x_1 + 2x_2 + 4x_3 + 50}{x_1 + 5x_2 + 5x_3 + 50} - \frac{x_1 + 2x_2 + 4x_3 + 50}{5x_2 + 4x_3 + 50} \\ s.t. \quad & 2x_1 + x_2 + 2x_3 \le 10, \\ & x_1 + 6x_2 + 2x_3 \le 10, \\ & 5x_1 + 9x_2 + 2x_3 \le 10, \\ & 9x_1 + 7x_2 + 3x_3 \le 10, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Obtain the optimal solution $x_1^* = 1.11111$, $x_2^* = 0$, $x_3^* = 0$.

Example 3.

$$\min \frac{37x_1 + 73x_2 + 50}{13x_1 + 13x_2 + 50} + \frac{13x_1 + 13x_3 + 50}{37x_1 + 73x_2 + 50}$$
$$s.t.\quad 5x_1 - 3x_2 = 3,$$
$$1.5 \le x_1 \le 3.$$

Set $\varepsilon = 10^{-6}$, obtain the optimal solution $x_1^* = 1.5$, $x_2^* = 1.5$.

Example 4.

$$\max \frac{3x_1 + 4x_2 + 50}{3x_1 + 5x_2 + 4x_3 + 50} - \frac{3x_1 + 5x_2 + 3x_3 + 50}{5x_1 + 5x_2 + 4x_3 + 50} - \frac{x_1 + 2x_2 + 4x_3 + 50}{5x_2 + 4x_3 + 50} - \frac{4x_1 + 3x_2 + 3x_3 + 50}{3x_2 + 3x_3 + 50}$$
$$s.t.\quad 6x_1 + 3x_2 + 3x_3 \le 10,$$
$$10x_1 + 3x_2 + 8x_3 \le 10,$$
$$x_1, x_2, x_3 \ge 0.$$

Obtain the optimal solution $x_1^* = 0$, $x_2^* = 3.33333$, $x_3^* = 0$.

Example 5.

$$\max \frac{37x_1 + 73x_2 + 13}{13x_1 + 13x_2 + 13} - \frac{63x_1 - 18x_2 + 39}{13x_1 + 26x_2 + 13} + \frac{13x_1 + 13x_2 + 13}{63x_2 - 18x_3 + 39} - \frac{13x_1 + 26x_2 + 13}{37x_1 + 73x_2 + 13}$$
$$s.t.\quad 5x_1 - 3x_2 = 3,$$
$$1.5 \le x_1 \le 3.$$

Obtain the optimal solution $x_1^* = 3$, $x_2^* = 4$.

Example 6.

$$\min \frac{3x_1 + 5x_2 + 3x_3 + 50}{3x_1 + 4x_2 + x_3 + 50} + \frac{3x_1 + 4x_2 + 50}{3x_1 + 3x_2 + 50} + \frac{4x_1 + 2x_2 + 4x_3 + 50}{4x_1 + x_2 + 3x_3 + 50}$$
$$s.t.\quad 2x_1 + x_2 + 5x_3 \le 10,$$
$$x_1 + 6x_2 + 2x_3 \le 10,$$
$$5x_1 + 9x_2 + 2x_3 \le 10,$$
$$9x_1 + 7x_2 + 3x_3 \le 10,$$
$$x_1, x_2, x_3 \ge 0.$$

Obtain the optimal solution $x_1^* = 0, x_2^* = 0$, $x_3^* = 0$.

Example 7.

$$\min \frac{3x_1 + 5x_2 + 3x_3 + 50}{3x_1 + 4x_2 + 5x_3 + 50} + \frac{3x_1 + 4x_2 + 50}{4x_1 + 3x_2 + 2x_3 + 50} + \frac{4x_1 + 2x_2 + 4x_3 + 50}{5x_1 + 4x_2 + 3x_3 + 50}$$
$$s.t.\quad 2x_1 + x_2 + 5x_3 \le 10,$$
$$x_1 + 6x_2 + 2x_3 \le 10,$$
$$5x_1 + 9x_2 + 2x_3 \le 10,$$
$$9x_1 + 7x_2 + 3x_3 \le 10,$$
$$x_1, x_2, x_3 \ge 0.$$

Obtain the optimal value $V^* = 3.0029$.

Numerical result shows that our algorithm can globally solve global optimization problem (GFP).

From numerical experiments, it is seen that computational efficiency of our algorithm is higher and can be used to large scale of linear sum of ratios problem GFP.

VI. Concluding Remarks

In this paper, we present a branch and bound algorithm for solving general linear fractional problem GFP. To globally solve problem GFP, we first convert it into an equivalent problem $\mathrm{EP}(H_0)$, then, through using linearization method, we obtain a linear relaxation programming problem of $\mathrm{EP}(H_0)$. In the algorithm, First, the branching process takes place in the space R^n rather than in the space R^n. This economizes the computation required to solve problem GFP. This mainly due to the fact that the numbers of ratios in the objective function of problem GFP is smaller than the number of decision variables n in the problem. Second, the upper bounding sub-problems are linear programming problems that are quite similar to one another. These characteristics of the algorithm offer computational advantages that can enhance the efficiencies of the algorithm.

It is hoped that in practice, the proposed algorithm and ideas used in this paper will offer valuable tools for solving general linear fractional programming.

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