An Improved Implementation of Preconditioned Conjugate Gradient Method on GPU

Yechen Gui
Shenzhen Institute of Advanced Technology, Chinese Academy of Sciences, Shenzhen, China
Email:liaoran919@yahoo.com.cn

Guijuan Zhang
Shenzhen Institute of Advanced Technology, Chinese Academy of Sciences, Shenzhen, China
Email:gj.zhang@siat.ac.cn

***Abstract*—An improved implementation of the Preconditioned Conjugate Gradient method on GPU using CUDA (Compute Unified Device Architecture) is proposed. It aims to solving the Poisson equation arising in liquid animation with high efficiency. We consider the features of the linear system obtained from the Poisson equation and propose an optimization method to solve it. First, a novel storage format called mDIA (modified diagonal storage format) is presented to improve the efficiency of the Sparse Matrix-Vector product (SpMV) operation. Second, a parallel Jacobi iterative method is proposed when using the Incomplete Cholesky preconditioner to explore inherent parallelism. Third, CUDA streams are also introduced to overlap computations among separate streams. The proposed optimization technique is embedded into our GPU based PCG algorithm. Results on Geforce G100 show that our SpMV kernel yields an improvement of nearly 100% for large sparse matrix with more than 30, 0000 rows. Also, a speedup of more than 7 is obtained for PCG method, making the real-time physics engine possible.**

***Index Terms*—CUDA, PCG, Incomplete Cholesky preconditioner, SpMV, Poisson equation**

I. INTRODUCTION

In the past few years, Graphics Processing Unit (GPU) has evolved into a unified powerful many-core processor. The modern GPUs are well suited for compute-intensive tasks and massively parallel computation (e.g., solving matrix problems [1] [2]). As one of the most common and important matrix problems, solving the large-scale linear system can be significantly accelerated if corresponding algorithms can be mapped well to the structure of the GPU and be accord with SIMD (Single Instruction, Multiple Data) pattern.

In this paper, we focus on the problem of solving the Poisson equation. The equation arises in many applications such as computational fluid dynamics, electrostatics, magnetostatics, etc. Numerical solution of the Poisson equation leads to a large sparse linear system. It is usually solved by iterative methods such as the best-known conjugate gradient (CG) method instead of direct methods (e.g., Gaussian elimination). The CG method can be easily implemented to solve linear systems that have a symmetric, definite positive (SPD) matrix [3]. However, it is often used with a suitable preconditioner in order to achieve high convergence rates in large scale applications. A CG algorithm with a preconditioner is called preconditioned conjugate gradient algorithm (PCG) and it has been proven to be efficient and robust in a wide range of applications [4].

Our goal is to solve Poisson equation efficiently by applying PCG algorithm on the Nvidia GPU architecture using CUDA [5]. Since the SpMV routine is the bottleneck of PCG algorithm that consumes nearly 80% of the total time, we present a novel storage format called mDIA storage format to optimize it. Moreover, we parallelize the traditional Jacobi iterative method to solve the lower Cholesky triangular equation when using the Incomplete Cholesky (IC) preconditioner. In addition, to effectively overlap the computation, CUDA streams are also adopted in this paper. Results show that our method obtains a speedup of 7 for PCG algorithm on Geforce G100.

The paper is organized as follows. The next section introduces the background of our method. The related work on GPU based PCG methods are reviewed first, and then we give a brief introduction of our linear system generated from Poisson equation. GPU architecture and our optimization algorithm based on GPU are presented in Section 3. In this section, the optimization techniques are discussed in detail. Section 4 shows experimental results followed by conclusions in Section 5.

II. BACKGROUND

2.1. Related Work

Jeff Bolz et.al [6] was the first to implement CG method on GPU using shader language and the speedup was about 1.5x. He also showed the feasibility of using the Compressed Row Storage Format for SpMV routine. After the advent of NVIDIA CUDA, GPU based iterative methods have been widely used to solve the sparse linear systems [7][8][9]. For example, Georgescu et.al [7] discussed how CG method could be aligned to the GPU architecture. They also discussed the problem with precision and applied different preconditioners to

doi:10.4304/jsw.7.12.2695-2702

accelerate convergence. In particular, they stated that for double precision calculations, problem having condition number less than 10 may converge and give a speedup also. In 2009, Buatois et al. [10] introduced their framework CNC for solving general sparse linear systems on GPU with a Jacobi-preconditioned CG method. Their method achieved a speedup of 3.2. However, they warned that GPU is only able to provide comparable accuracy because as the iterations increase, the precision drops in comparison to CPU. They also exploited some of the techniques like register blocking to optimize their algorithm. In [11], the CG method with Incomplete Poisson preconditioning was implemented on a multi-GPU platform. It mainly focused on overlapping communication between different GPUs by interactively exchanging boundary stream and inner stream. Their results showed that the performance can grow proportionally to the problem size and showed a good scalability. In work published by A.Asgasri[9], the author parallelized a Chebyshev polynomial preconditioner to improve the performance of PCG method based on GPU.

As for the CG algorithm, nearly 80% of the total time is consumed by SpMV routine. It yields only a small fraction of the machine peak performance due to its indirect and irregular memory access. Therefore, there exists a large amount of work focusing on speeding up SpMV routine. Typical methods often use CSR (Compressed sparse row) format, COO (the coordinate) format and the DIA (diagonal) storage formats to mitigate the irregularity [12]. In a recent study by Nathan Bell [13], a hybrid method that used the modified ELL-COO format to store the sparse matrix delivered high throughput. However, it relied on an additional sweep operation to find out the number of nonzero elements in the matrix.

2.2. The Linear System Derived from the Poisson Equation

The Poisson equation in liquid animation is a second-order PDE as shown in equation (1). It is used to compute the pressure p.

$$\nabla^2 p = \frac{\rho}{\Delta t}\nabla u \tag{1}$$

Note that in the Poisson equation (1), p is the pressure, ρ is the density and Δt is time step. It can be further transformed into equation (2) according to finite difference method.

$$\begin{aligned} &-p_{i-1,j,k}-p_{i,j-1,k}-p_{i,j,k-1}+6p_{i,j,k}-p_{i+1,j,k}-p_{i,j+1,k}-p_{i,j,k+1} \\ &=-\Delta x^2\frac{\rho}{\Delta t}\left(\frac{u_{i+1,j,k}-u_{i,j,k}+v_{i,j+1,k}-v_{i,j,k}+w_{i,j,k+1}-w_{i,j,k}}{\Delta x}\right) \\ &=-\Delta x\frac{\rho}{\Delta t}(u_{i+1,j,k}-u_{i,j,k}+v_{i,j+1,k}-v_{i,j,k}+w_{i,j,k+1}-w_{i,j,k}) \end{aligned} \tag{2}$$

In equation (2), Δx is space interval, u = (u,v,w) is the velocity field. Let

$$b=-\Delta x\frac{\rho}{\Delta t}(u_{i+1,j,k}-u_{i,j,k}+v_{i,j+1,k}-v_{i,j,k}+w_{i,j,k+1}-w_{i,j,k}). \tag{3}$$

Equation (2) can be converted into $Ap = b$, and our goal is to solve the unknown vector p.

It can be proved that A is a sparse, positive-definite and symmetry matrix. In addition, we also explore some other features of matrix A in order to design more efficient algorithms. See the left side of equation (2), each row of A has no more than 7 nonzero elements. In this row, the diagonal element is a nonzero integer while the other nonzero elements equal to -1. Other rows also have similar structures. All these features will be considered in our new algorithm. Details will be given in the following sections.

2.3. The PCG Algorithm

Consider

$$Ax = b, \tag{4}$$

where x is an unknown vector, b is a known vector, A is a known SPD matrix. According to PCG algorithm, equation (4) can be written as

$$M^{-1}Ax = M^{-1}b, \tag{5}$$

where matrix M is a preconditioner[4].

Given the inputs A, b, a starting vector x, a preconditioner M, a maximum number of iterations k_max and a error tolerance err, the PCG algorithm can be described in fig. 1. In this figure, a set of α-orthogonal search directions $\alpha_0, \alpha_1, \alpha_2 \cdots \alpha_n$ are constructed by the conjugation of the residues $r_0, r_1, r_2 \cdots r_n$ respectively. Then in the k th iteration step, x_k takes exactly one step of the length h_k along the direction a_k. If the convergence conditions $err < \varepsilon$ and $k < \max\_k$ are met, the iterative process is terminated. Note that in our method, we set $x_0 = 0$.

To compute $M^{-1}r_k$ at each iteration step, we choose IC preconditioner in our method to improve the convergence rate [4]. An IC preconditioner can be obtained by factoring a matrix A into the form LL^T where L is a lower Cholesky triangular matrix. L is restricted to have the same pattern of nonzero elements as A and other elements of L are thrown away. Therefore, M equals LL^T and $z_k \leftarrow M^{-1}r_k$ can be converted into $(LL^T)z_k = r_k$. As a result, z_k can be directly computed by forward and then backward substitutions.

III. IMPLEMENTATION ON GPU

3.1 *GPU Architecture and CUDA*

The new generation of GPU adopts the unified shader architecture CUDA and promise up to 900 Gflops(single precision) of computational power.

A GPU can be seen as a SIMD processer. What this means is that there are an army of processers executing the same instructions in parallel independently. Take fig. 2 for example, a GPU has a scalable array of multithreaded Streaming Multiprocessors (SPs). Each multiprocessor creates, manages, schedules, and executes groups of 32 parallel threads which are called warps. Individual threads composing a warp start together at the

$$
\begin{aligned}
&r_0 \leftarrow b - Ax_0 \\
&z_0 \leftarrow M^{-1} r_0 \\
&h_0 \leftarrow z_0, \\
&err_{old} \leftarrow < r_0, z_0 > \\
&err_{new} \leftarrow err_{old} \\
&While\ (\mathrm{err}_{new} < \varepsilon \ \ || \ \mathrm{k} < \mathrm{k\_max}) d \\
&\alpha_k \leftarrow \frac{err_{new}}{< h_{k-1}, Ah_{k-1} >} \\
&x_k \leftarrow x_{k-1} + \alpha h_{k-1} \\
&r_k \leftarrow r_{k-1} - \alpha Ah_{k-1} \\
&z_k \leftarrow M^{-1} r_k \\
&err_{old} \leftarrow err_{new} \\
&err_{new} \leftarrow < r_k, z_k > \\
&\beta \leftarrow \frac{err_{new}}{err_{old}} \\
&h_k \leftarrow z_k + \beta h_{k-1}
\end{aligned}
$$

Figure 1. PCG algorithm.

same program address, but have their own instruction address counter and register state. Therefore they are free to branch and execute independently.

On GPU chip, each multiprocessor has a set of memories associated with it. They are: on-chip shared memory, global memory, read-only constant cache, and read-only texture cache. Among these memories, global memory is the biggest in size but with highest access latency. On the other hand, Shared memory, constant cache as well as the texture cache resides on chip and can be accessed more efficiently. Note that shared memory is only visible to one block and threads of other block cannot access the data stored in it.

The scalable characteristic of modern GPU provides coarse-grained and the fine-grained data parallelism. They guide the programmer to partition the problem into sub-problems that can be solved independently in parallel by blocks of threads. Moreover, each sub-problem can be divided into finer pieces that can be solved cooperatively in parallel by all threads within the block. Fig. 3 gives an example. In this figure, kernels are launched on GPU device and executed by multiple equally-shaped thread blocks.

3.2 Overview

Algorithm 1 shows our framework for PCG implementation on GPU. In this framework, we use two kernels to complete the computation involved in the *for* loop. They are the SpMV kernel that computes matrix-vector multiplication such as Ah_{k-1}, the preconditioning kernel that computes $M^{-1}r_k$.

Besides the above kernels, other operations such as dot products among vectors can be done efficiently by cublas library [14] because they all belong to level-1 BLAS (Basic Linear Algebra Subprograms) functions.

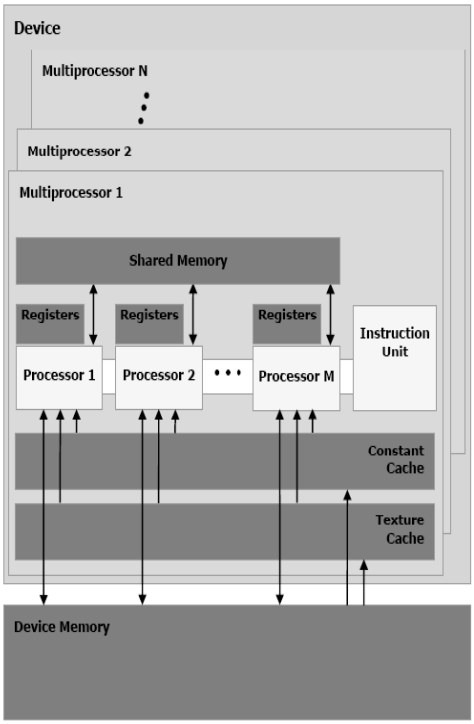


Figure 2. GPU architecture.

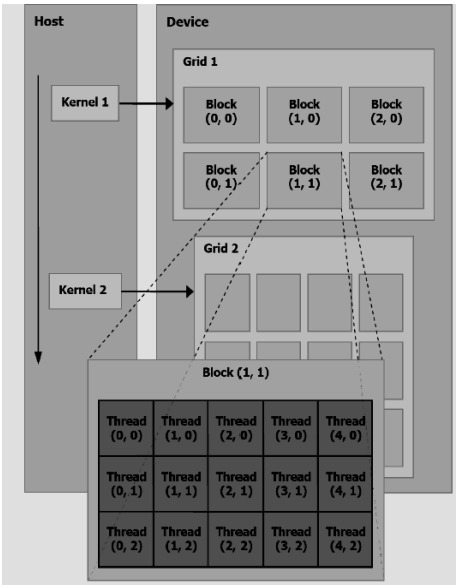


Figure 3. Serial code executes on the host while parallel code executes on the device.

In order to improve the efficiency of our GPU based PCG algorithm, we focus on two most expensive kernels here: the SpMV kernel and the preconditioning kernel. In addition, to get a higher level of concurrency, we use a technique called streams [15] to run independent kernels asynchronously so as to overlap the computation.

Algorithm 1. Algorithm for our PCG method on GPU

//Use *cublas*() to compute dot products
//and then obtain err
$r_0 \leftarrow b - Ax_0$
$z_0 \leftarrow M^{-1} r_0$
$h_0 \leftarrow z_0$,
$err_{old} \leftarrow < r_0, z_0 >$
$err_{new} \leftarrow err_{old}$
while ($err_{new} < \varepsilon$ && k<k_max)*do*
//*spmv* kernel *for* calucating Ah_{k-1}
kenel _ spmv()
//Use *cublasdot*()

$$\alpha \leftarrow \frac{err_{new}}{< h_{k-1}, Ah_{k-1} >} \quad (1)$$

//Use *cublasSaxpy*()

$$x_k \leftarrow x_{k-1} + \alpha h_{k-1} \quad (2)$$

$$r_k \leftarrow r_{k-1} - \alpha Ah_{k-1} \quad (3)$$

//*P*reconditioning kernel
$z_k \leftarrow M^{-1} r_k$
// Devcie *assignment*
$err_{old} \leftarrow err_{new}$
//Use *cublasdot*()

$$err_{new} \leftarrow < r_k, z_k > \quad (4)$$

//Use *cublasSaxpy*()

$$\beta \leftarrow \frac{err_{new}}{err_{old}}$$

$$h_k \leftarrow z_k + \beta h_{k-1} \quad (5)$$

$k++$
end while

3.3 SpMV Kernel

Sparse Matrix-vector multiplication (SpMV) is one of the most fundamental and important operations in sparse matrix computations. It is the dominant cost in many iterative methods for solving large-scale linear systems. Recently, several research groups have reported their implementation on CUDA–compatible GPUs and show that the storage pattern such as CSR, ELL, HYBRID formats can be efficiently accessed by CUDA threads. However, except for the CSR format, all other formats have to fill in zeros to keep the array strictly aligned, thus causing memory waste.

Since the SpMV kernel with CSR format provides important insight for understanding our algorithm, we discuss the implementation of SpMV GPU kernels with CSR and mDIA respectively in the following subsections.

3.3.1 SpMV with CSR format

CSR format is one of the most popular sparse matrix representations. In this format, an N-by-N sparse matrix with K nonzero elements is stored as two arrays: one array *val* holds the K nonzero elements and the other array *col* holds the column indexes of these nonzero elements. What's more, an additional array *Rowptr* with the length N+1 is used. The first N components of *Rowptr* record the indexes of the first element in each row while the last one denotes the number of nonzero elements in the matrix. Fig. 4 gives an example. Unlike the ELL or DIA, CSR format doesn't waste any memory space.

$$Matrix = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 9 & 7 & 0 & 8 & 2 \\ 0 & 6 & 1 & 0 & 0 \\ 1 & 0 & 3 & 9 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$val = [1\ 3\ 9\ 7\ 8\ 2\ 6\ 1\ 1\ 3\ 9\ 2]$$

$$col = [0\ 1\ 0\ 1\ 3\ 4\ 1\ 2\ 0\ 2\ 3\ 2]$$

$$Rowptr = [0\ 2\ 6\ 8\ 11\ 12]$$

Figure 4. CSR format for sparse Matrix.

To parallelize the SpMV operation with CSR format, a scheme called scalar CSR kernel [13] is used. In this kernel, one thread is used to fetch one row of the matrix A and then complete the dot product for one component of the result vector. The computations of all threads are independent. The data parallelism as well as the access pattern of scalar CSR kernel is shown in Fig. 5. It gives a simplified example of the allocation of the threads, in which the array data, Col Index, and Rowptr are stored in global memory for the dot product operation.

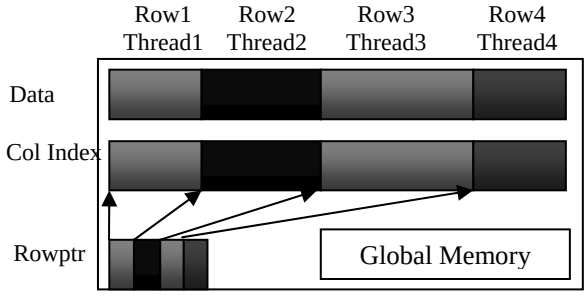


Figure 5. Scalar SpMV Kernel with CSR format.

3.3.2 SpMV with mDIA format

(1) Our mDIA storage format

Our matrix has a regular pattern that the elements off the main diagonal are all assigned to a constant integer -1while the nonzero diagonal elements are also integers. The number of nonzero elements per row varies from 3 to 7 and this irregularity makes ELL or hybrid pattern infeasible, since they will cause large zero fill-ins.

In mDIA format, the constant value is stored in the constant memory. As a result, 1D array Diag only needs to store the diagonal elements. Because all diagonal elements in our matrix are nonzero values, the row and

column indexes of them can be easily obtained from *Diag*. Array *col* only needs to record column indexes of the constant value. Besides, Array *Rowptr* is used to tell where a new row begins in the array *col*, similar to array *Rowptr* in CSR format. Fig. 6 shows a portion of one matrix and its corresponding storage mechanism. Note that the constant -1 is stored in the constant memory.

Since most of the nonzero elements are off the main diagonal in our matrix and they are stored in the constant memory, the memory usage can be significantly reduced compared with the CSR format. Also, the constant memory, a small high-speed cache residing in the global memory on GPU, enables us to fetch data efficiently [15].

$$Matrix = \begin{bmatrix} 6 & -1 & 0 & 0 & 0 \\ -1 & 7 & 0 & -1 & 0 \\ 0 & -1 & 6 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$diag = [6 \;\; 7 \;\; 6 \;\; 3 \;\; 2]$$

$$col = [1 \;\; 0 \;\; 3 \;\; 1 \;\; 0 \;\; 2 \;\; 4 \;\; 2]$$

$$Rowptr = [0 \;\; 1 \;\; 3 \;\; 4 \;\; 7 \;\; 8]$$

device _constant_ val =-1

Figure 6. mDIA storage format.

(2) The SpMV kernel

To implement the SpMV kernel with mDIA format on GPU, we assign one thread to compute one component of the result vector. Take thread *i* for example. It computes the dot product between the i-th row of our matrix and the vector. First it fetches the diagonal nonzero element from *diag[i]*. Then the column indexes of the constant elements from *col[Rowptr[i]]* to *col[Rowptr[i+1]]* are read contiguously. Finally, the dot products operation is executed.

Considering that the vector is reused in the computation of the dot product, we bind it to a 1D texture. This can bring potentially higher bandwidth and can be used to avoid uncoalesced loads from global memory [15]. Other array such as *col* are stored in the global memory. Fig. 7 presents the pseudo-code of our implementation.

3.4 Preconditioning Kernel using Jacobi Method

Another important kernel in our PCG algorithm is the preconditioning kernel. According to IC preconditioner, $z_k \leftarrow M^{-1}r_k$ is converted into $(LL^T)z_k = r_k$ where L is the lower Cholesky triangular matrix. Thus, z_k can be obtained by solving $L(L^T z_k) = r_k$ in two stages: forward substitution $Ly = r_k$ and backward substitution $L^T z_k = y$.

However, the direct method of backward and forward substitution cannot be used to solve all the components simultaneously on GPU because the computation of the ith component of z_k relies on all its previous components.

To get a higher level of parallelism, we use Jacobi method instead of direct method in this paper. Jacobi iterative method is data independent that can be well aligned to SIMD pattern and improves parallelism significantly.

```
__Constant__ val = -1;
__ global__ void SpMV(float  *diag, float *col, float
*Rowptr, float *x, )
{
   int thread_id = blockDim.x * blockIdx.x + threadIdx.x;
   int grid_size = gridDim.x * blockDim.x;
   for(int row = thread_id; row < num_rows; row +=
       grid_size)
     {
          const int row_beg = Rowptr[row];
          const int row_end   = Rowptr[row+1];
          float sum = 0;
          for (int jj = row_beg; jj < row_end; jj++)
          {
              sum += val*fetch_x(Col[jj], x, UsedTex);
              sum += diag[row] *fetch_x(row, x, UsedTex);
          }
          y[row] = sum;
     }
}
```

Figure 7. Pseudo-code of our SpMV kernel

3.4.1 Jacobi iterative method

Jacobi iterative method is a numerical solution of a system of linear equations with largest absolute values in each row and column dominated by the diagonal element.

To solve our lower Cholesky triangular equation $Ly = r_k$, we first decompose the lower Cholesky triangular matrix L into a diagonal matrix D and a lower triangle matrix R. Then the system of linear equations becomes

$$(D + R)y = r_k, \tag{6}$$

and finally

$$Dy = r_k - Ry. \tag{7}$$

Therefore, y can be solved iteratively by

$$y_{k+1} = D^{-1}(r_k - Ry_k). \tag{8}$$

Next, z_k can be computed by solving upper Cholesky triangular equation $L^T z_k = y$ in the same way.

3.4.2 Parallel Jacobi algorithm

Fig. 8 (a) shows the iterative process of our parallel Jacobi algorithm. In this figure, *d_old* stores the result from the previous step and *d_new* is used to update the current computation. The constant vector *d_const* = $D^{-1}r_k$, where the diagonal matrix D is stored in the array d_diag. d_R stores the lower triangular matrix R and d_res is used to compute the residue. Note that most

of the operations such as $D^{-1}b$ can be converted to level-1 BLAS operation among the vectors. These operations could be easily parallelized using CUDA. Fig. 8 (b) shows the kernel named *VecDiv_kernel* for computing $D^{-1}b$ and the kernel named *VectorSub_kernel* for computing the substraction betweeen two vectors.

```
while(err < tol && k < k max)
{
// d_yNew ← Ry_k
SpMV(d_R, d_yOld, d_yNew);
// d_yNew ← D^{-1}(Ry_k)
VecDiv_kernel(d_diag, d_yNew, N);
// d_yNew ← D^{-1}(Ry_k) + D^{-1}r_k
cublasSaxpy(N, 1.0, d_const, 1, d_yNew, -1);
// d_res ← residue
VectorSub(d_yNew, d_yold, d_res, N);
err = cublasSasum(N, d_res, 1);
// d_yOld ← d_yNew
cublasScopy(N, d_yNew, 1, d_yOld, 1);
k++;
}
```

(a) The iterative process.

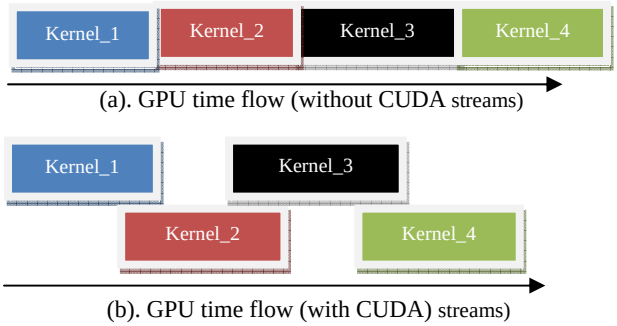
```
__global__ void VectorSub_kernel(float* A, float*
B, float* C,int N)
{
        int i = blockDim.x * blockIdx.x +
        threadIdx.x;
        float a  = 0.0f;
        if (i< N)
          {
            a = A[i] - B[i];
            C[i] = a;
          }
}


__global__ void VecDiv_kernel1(float* A, const
float* B, float* C,int N)
{
        int i = blockDim.x * blockIdx.x +
                threadIdx.x;
        float a  = 0.0f;
        if (i< N)
        {
                a = __fdividef(A[i],B[i]);
                C[i] = a;
         }
}
```

(b) Two GPU kernels involved in this process.

Figure 8. Parallel Jacobi iterative method.

3.5 Parallelism with Streams

CUDA applications can manage concurrency through streams. A stream is a sequence of commands that execute in order. Different streams, on the other hand, may execute their commands out of order concurrently. Fig. 7 illustrates the GPU time flow for sequential (Fig. 9 (a)) and concurrent (Fig. 9 (b)) kernel executions.



(a). GPU time flow (without CUDA streams)

(b). GPU time flow (with CUDA) streams)

Figure 9. GPU Time flow with CUDA streams

We adopt this optimization technique for our two independent tasks as shown in fig. 10.

```
SetKernelStream(streams[0]);
x_k ← x_{k-1} + αh_{k-1} ;
SetKernelStream(streams[1]);
r_k ← r_{k-1} − αAh_{k-1};
```

Figure 10. The operations using CUDA streams.

As a result, the computation of the two tasks can be overlapped and the GPU resources can be used more effectively.

IV. Results

We use Geforce G100 to test the performance of our parallel PCG algorithm. Geforce G100 has 8 CUDA cores and the peak performance for single precision is 10.4 Gflops. The CPU used is AMD 7750 dual-core processers with the core frequency of 2.7GHz. The CPU implementation of PCG is single-threaded.

Table 1 shows some matrices that generated from our Poisson equation. They will be used in our performance test. In this section, we start from testing the performance of our SpMV kernel and then test Jacobi iterative method for solving the lower Cholesky triangular equations, followed by CUDA stream results.

TABLE.1
MATRICES USED FOR EXPERIMENTS

MATRIX	#N	#Nonzeros
Matrix_1	8087	49915
Matrix_2	22028	131,118
Matrix_3	65043	394,877
Matrix_4	140,120	876,518
Matrix_5	209,908	1,325,066
Matrix_6	274,949	1,755,263
Matrix_7	304,207	1,962,311

4.1.SpMV Kernel Test

4.1.1 SpMV kernels performance

Table 2 shows the performance of our SpMV kernel against the SpMV kernel with CSR format. In this table, *GPU Time* accounts for the total time consumed for the

SpMV kernel in our PCG algorithm and the speedup is the ratio compared with *CPU Time*. On CPU, the SpMV routine is implemented using CSR format. According to the results, our SpMV kernel runs an average of one time faster than that in CSR format and it offers an average of about 10 times speedup compared with the CPU version.

TABLE 2
GPU/CPU SPMV KERNELS PERFORMANCE (SECONDS)

matrix	Timings for spMV CPU routine	Timings for spMV GPU kernel (seconds)		Speedup	
		CSR	mDIA	CSR	mDIA
Matrix_1	0.48	0.09	0.05	5.4	9.0
Matrix _2	1.34	0.25	0.14	5.4	9.5
Matrix _3	5.50	1.00	0.57	5.5	9.6
Matrix _4	15.15	2.63	1.52	5.8	10
Matrix _5	29.67	5.34	2.99	5.6	9.9
Matrix _6	44.82	8.03	4.50	5.5	9.96
Matrix _7	54.69	9.50	5.41	5.1	10.8

4.1.2 The performance of CG&PCG algorithm with our SpMV kernel

We embed our SpMV kernel into CG&PCG method On GPU. Here, the IC preconditioner of PCG method is solved by direct method. Further improvement using Jacobi iterative method will be presented in the next subsection.

We compare our results with CG&PCG methods using CSR format as shown in Table 3. It illustrates that due to our SpMV kernels, the CG algorithm outperforms by nearly 50% when the number of nonzero elements reached 1,962,311.

However, when PCG method is applied, this table shows that our advantage over the CSR_based method has been less obvious; and when the dimension of matrix has reached up to 304,207, the performance improvement is only about 10%. This is due to the sequential nature of the direct method as mentioned before.

4.2 Jacobi Method for IC Preconditioner

In this section, we adopt two variants: the direct method and the Jacobi iterative method to explore GPU performance for PCG algorithm.

TABLE 3.
TIME COST FOR CG AND PCG ALGORITHM IMPLEMENTED ON GPU RESPECTIVELY (SECONDS)

matrix	CG steps	Timings for GPU based CG method(seconds)		PCG steps	Timings for GPU based PCG method(seconds)	
		mDIA	CSR		mDIA	CSR
Matrix_1	84	0.64	0.68	17	0.50	0.50
Matrix_2	90	0.76	0.86	21	1.15	1.17
Matrix_3	123	1.40	1.80	23	2.63	2.71
Matrix_4	146	2.42	3.69	25	4.8	5.09
Matrix_5	192	4.44	6.70	30	7.9	8.32
Matrix_6	221	6.41	9.87	34	10.98	11.87
Matrix_7	234	7.25	11.53	35	11.35	12.77

When solving the lower Cholesky triangular equation, Timings for the parallel Jacobi iterative method and the direct method are both accumulated at every iterative step and their final costs are showed in Table 4. From this table, it can be noticed that GPU performance of the Jacobi iterative method has been efficiently improved about 3 times.

TALBE 4.
TIME COST FOR GPU SOLVER OF JACOBI METHOD AND DIRECT METHOD WHILE APPLYING IC PRECONDITIONER

matrix	Timings for Jacobi solver (seconds)	Timings for direct solver (seconds)
Matrix_2	0.18	1.001
Matrix_3	0.50	2.533
Matrix_4	1.225	3.912
Matrix_5	2.288	6.030
Matrix_6	3.377	9.454
Matrix_7	3.893	11.021

4.3 The Speedup of Our GPU based PCG Algorithm

Fig. 11 shows the speedup of our parallel PCG algorithm after using our SpMV kernel, Jacobi iterative method, as well as CUDA streams. It can be seen from the figure that there has been at least 16% increase for our PCG algorithm compared that with CSR formats. Furthermore, our PCG algorithm with three optimization techniques proposed obtains an average of 6 times speedup, while the CSR method only offers an average of 4.

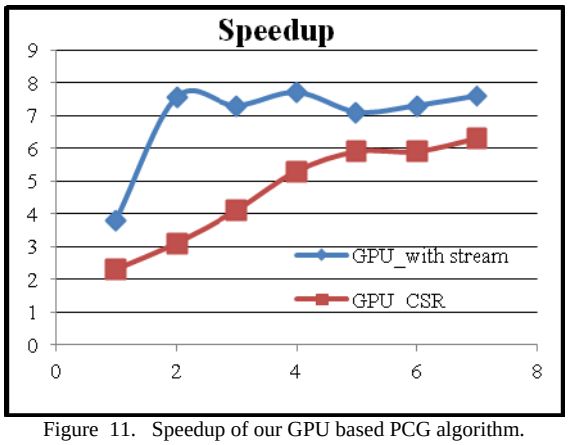


Figure 11. Speedup of our GPU based PCG algorithm.

V. CONCLUSIONS

In this work, we propose an optimization method for GPU based PCG algorithm. It is designed to solve the Poisson equation arising in liquid animation efficiently. By utilizing optimized SpMV kernel, iterative Jacobi method and CUDA streams, our method improves the efficiency of solving large sparse linear systems significantly. Experimental results also show the effectiveness of our method.

Next we will focus on seeking out the other potential bottleneck of PCG algorithm by CUDA Visual Profiler. Besides, studies on finding more suitable preconditioners for GPU-based PCG algorithm should be taken.

ACKNOWLEDGMENT

The authors wish to thank Prof. Shengzhong Feng. This work was supported in part by the National High-Tech Research and Development Plan of China (863 program) under Grant No.2009AA01A129-2, the Science

and Technology Project of Guangdong, Province under the grant No.2010A090100028, National Natural Science, Foundation of P. R. China under the Grant No.60903116，and Knowledge, Innovation Project of the Chinese Academy of Sciences, No.KGCX2-YW-131，the Science and Technology Project of Shenzhen, under the grant No. JC200903170443A, ZD201006100023A.

VI. REFERENCE

[1] J.D.Hall,N.A. Carr and J.C.Hart,"Cache and Bandwidth aware matrix multiplication on the GPU,",2003.UIUC Technical Report UIUCDCSR-2003-2328(2003)

[2] N.Galopppo, N.K. Govindaraju,"LU-GPU:Efficient algorithms for solving dense linear systems on graphics hardware," In SC'05: Proceedings of the 2005 ACM/IEEEE conference on Supercomputinng, page3, Washington D.C, USA, 2005, IEEE Computer Society. ISBN1-59593-061-2

[3] Kendell A. Atkinson (1988), *An introduction to numerical analysis* (2nd ed.), Section 8.9, John Wiley and Sons.

[4] A.V. Knyazev, I. Lashuk, Steepest descent and conjugate gradient methods with variable preconditioning, SIAM J. Matrix Analysis and Applications 29(4), 1267-1280, 2007.

[5] Nvidia CUDA. Website,2009, http://www.nvidia.com/cuda.

[6] J. Bolz, I. Farmer, E. Grinspun, and P. Schr¨ooder, "Sparse matrix solvers on the GPU: Conjugate gradients and multigrid," in SIGGRAPH '03:ACM SIGGRAPH 2003 Papers, 2003, pp. 917–924.

[7] W.A.Wiggers, V.Bakker, A.B.J.Kokkeler and G.J.M.Smit, "Implementing the conjugate gradient algorithm on multi-core systems," In J.Nurmi,J.Takala and O.Vainio, editors, Proceedings of the International Symposium on System-on-Chip, Tampere, pages 11-14, Piscataway, NJ,November 2007,IEEE,ISBN 1-4244-1367-2

[8] Maringanti.A.;Athavale.V.; Patkar.S.B.; "Acceleration of Conjugate Gradient Method for cirruit simulation using CUDA" In High Performance Computing (HiPC), 2009 International Conference, Kochi, pp438-444

[9] A.Asgasri, J.E.Tate. Implementing the Chebyshev Polynomial Preconditioner for the iterative solutions of linear systems on massively parallel graphics processors http://www.ele.utoronto.ca/zeb/publications /,2009

[10] L. Buatois, G. Caumon, and B. Levy, "Concurrent number cruncher: a GPU implementation of a general sparse linear solver," Int. J. Parallel Emerg. Distrib. Syst., 24(3):205–223,2009. ISSN 1744-5760

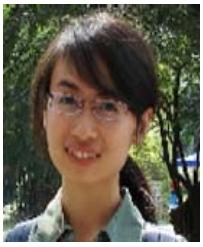
[11] Marco Ament, Gunter Knittel, Daniel Weiskopf, Wolfgang Strasser, "A Parallel Preconditioned Conjugate Gradient Solver for the Poisson Problem on a Multi-GPU Platform," pdp, pp.583-592, 2010 18th Euromicro Conference on Parallel, Distributed and Network-based Processing, 2010

[12] S. Williams, L. Oliker, R. Vuduc, J. Shalf, K. Yelick, and J. Demmel, "Optimization of sparse matrix-vector multiplication on emerging multicore platforms," in Proceedings of the 2007 ACM/IEEE conference on Supercomputing, 2007, pp. 1-12.

[13] Nathan Bell "Implementing Sparse Matrix-Vector Multiplication on Throughput-Oriented Processors" in *"Proc. Supercomputing '09"*, November 2009

[14] Nvidia, "CUDA Toolkit 4.0 CUBLAS Library" NVIDIA Corporation, Santa Clara, April, 2011

[15] Nvidia,"CUDA Programming Guide 4.0" NVIDIA Corporation, Santa Clara, April, 2011



Born in Jiangxi province in China, **Yechen Gui** is a graduate of Xidian University, where she earned her bachelor degree in 2006, majoring in biomedical engineering. After that, she went to Southern Medical University for her postgraduate studies. During that time, she became fairly interested in parallel algorithms in medical image processing. She implemented parallel ray-casting algorithm as well as CT reconstruction algorithms on GPU using CUDA and published two articles in two different core journals.

After **Yechen** gained her master degree in 2009, she worked as a research assistant in Shenzhen Institute of Advanced Technology, Chinese Academy of Sciences until now. During this time, she published 4 articles in parallel algorithms on GPU in all, one of which is accepted in the conference of 2010 GPU Solutions to Multi-scale Problems in Science and Engineering (GPU-SMP'2010), others of which are all indexed by EI. Now she's research field is towards computational fluid animations as well as parallel algorithms.