Multifractal Analysis and Modeling of Chaotic Channels

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Abstract—**This paper introduces multifractal theory into the research of chaotic channels, and a novel model of chaotic channel is established from nonlinear science standpoint. By studying the relationship between multifractal dimensions and work environments, increments probability distribution, and statistical self-similarity of low power line communication (L-PLC) channel which is a typical chaotic channel, we find that multifractal dimensions are key parameters for describing the chaotic channel, and reveal the inherent relationship between the chaotic channel and Fractional Brownian Motion (FBM) model. Finally, the multifractal model of chaotic channel based on FBM theory is established and applied to reconstructing received signals accurately.**

Index Terms—**multifractal, chaos, modeling, Fractional Brownian Motion**

I. INTRODUCTION

Due to frequency attenuation, noise interference, multipath interference, etc, communication channels usually show time-varying, and randomness. It is difficult to analyze communication channels quantitatively and model the channels effectively. Reference [1] analyze low power line channel and channel modeling. But it discusses five kinds of noises models respectively. In fact, these noises act on channel together. Only a general analysis of these elements will describe channel characteristics synthetically. Reference [2] presents a multipath model for multipath channel, which uses linear method to imitate the complex channel. And the method can only describe the channel within specific limits. Actually, communication channels show not only unstable, but also inherent regularity. The phenomenon is similar with definition of chaos. As a matter of fact, many researches indicate that chaos are common in communication channels, such as low voltage power line communication (L-PLC) channel, multipath channel, wireless channel, etc [3-11]. Chaos is subject associated with the discipline of nonlinear dynamics: the study of systems that respond disproportionately to stimuli. It is in accord with research of fractal. A fractal consists of geometric fragments of varying size and orientation but similar shape. Fractal structures are often the remnants of chaotic nonlinear dynamics. Wherever a chaotic process has shaped an attractor, fractals are likely to be the attractor trajectories [12]. For the tight relationship between chaos and fractal, multifractal theory is introduced into the study of chaotic channels.

We take L-PLC channel for example to research. Reference [3] has verified L-PLC channel is the result evolved by nonlinear chaotic system. In other words, L-PLC channel is a typical chaotic channel. In this paper, a novel model of chaotic channel is established, based on Fractional Brownian Motion (FBM) model. By statistical analysis of observed L-PLC signal, we research multifractal characteristics, increments probability distribution, and statistical self-similarity of L-PLC channel. Then we discuss the relationship between multifractal dimensions and work environments, and reveal the inherent relationship between the chaotic channel and FBM model. At last the multifractal model of chaotic channel is established with multifractal parameters calculated and applied to reconstructing received signals accurately.

II. MULTIFRACTAL DIMENSIONS

Geometrical properties of natural structure are the main areas of research for fractal theory which describes nonlinear dynamics mechanism of natural structure with a serial of fractal parameters. Scale invariability is the most important characteristic of fractal theory, which means that there are similar properties among various scales. Fractal theory already has been applied to expressing perplexing figures and processes [13-15]. However, monofractal dimension is only a unitary and average description of investigated subject, which is unable to reflect comprehensive and subtle information of different fractal structures caused by diverse areas, levels and local conditions. In this way, monofractal theory can not reveal the dynamic process of fractal structure. Therefore, concept of multifractal was proposed recently.

Multifractal is an infinite set composed of singular measure of diverse scaling exponents, such as generalized fractal dimension, generalized Hurst parameter, and multi-fractal spectrum, etc. It depicts local scaling property of different distributions. Multifractal theory is,

as it were, popularization of simple fractal dimension. Multifractal dimensions can be used to describe the space probability distribution of geometrical graphs or physical parameters and express the complexity of investigated subject [16-18]. Larger dimension means subject is more complicated. They can give more information than monofractal dimension. Actually, simple fractal dimension is only a point in multi-fractal spectrum. Due to all of these merits, multifractal is often applied to describing distribution properties of physical parameters with self-similarity characteristic. Moreover, multifractal supplies an effective mathematical model which describes local singular properties and complicated local structure of signals.

Definition of multifractal is as follows.

Consider a unit interval with unit mass. Separate the interval into N subintervals. The length of each subinterval is δ, and weight of the k th subinterval is μ_k. In this way, $\{\mu_k, k \geq 1\}$ stands for a random process. Define $\alpha(t_0)$ as a singular exponent of μ_k at t_0, and $\alpha(t_0)$ can be expressed as

$$\alpha(t_0) \equiv \lim_{a \to 0} \frac{\lg(\mu_k)}{\lg a} \qquad (1)$$

Where μ_k is the weight of subinterval at t_0. If above-mentioned limitation is not existed, then the singular exponent can not be defined. If series singular exponent $\alpha(t)$ of $\{\mu_k, k \geq 1\}$ varies as t changes, then it can be said that series $\{\mu_k, k \geq 1\}$ has multifractal property.

For received L-PLC signal, firstly, divide the signal into N voltage amplitude sets. Then the received signal is expressed as $V = \{(v, f(v)) : v \in \Delta_i, i = 1, 2, \ldots N\}$ where v is voltage amplitude of received signal, and Δ_i is amplitude section of the i th point of received signal, and $f(v)$ is the probability of signal point fallen into Δ_i.

Secondly, suppose if i th subinterval calibration is r, and the number of amplitudes of received signal fallen into this subinterval is $N_i(r)$ during observation, then the multifractal dimension can be expressed as

$$D_q = \begin{cases} \dfrac{1}{q-1} \lim_{r \to 0} \dfrac{\log \sum_i v_i^q(r)}{\log(r)}, & q \neq 1 \\[4ex] \lim_{r \to 0} \dfrac{\sum_i v_i(r) \log v_i(r)}{\log(r)}, & q = 1 \end{cases} \qquad (2)$$

Where $v_i(r) = N_i(r) / \sum_j N_j(r)$, and q is probability weighing exponent. As q is varied, different amplitude distribution of signal will determine D_q. $D_{+\infty}$ corresponds to the region where the points are mostly concentrated, while $D_{-\infty}$ is determined by the region where the points have the least probability to be found. D_0 has nothing to do with the probability $v_i(r)$. Therefore, through process

of weighing a complicated subject is separated into several regions with different singularity degrees.

If D_q and r satisfy simple logarithm straight relationship with fixed q and various r, it can be said that the signal has property of multifractal.

In L-PLC chaotic channel, influences caused by different environments are distinct. Moreover, the influences will affect the amplitude distribution of received signal. Suppose if r^{D_q} is calibration of points fallen into one amplitude set, then D_q is the number of these points during a limited period of time. Multifractal analysis of chaotic L-PLC signal can reveal evolutionary processes that signal experiences complicated nonlinear affection, such as frequency attenuation, noise interference, and multipath interference, etc. In this way, analysis of multifractal can depict the substantive characteristics of L-PLC signal. Actually, multifractal dimension spectrum $D_q \sim q$ includes all dimensions involved fractal theory. For instance, D_0 is known as the capacity dimension, D_1 as the information dimension, D_2 as the correlation dimension. Therefore, an appropriate $D_q(q \mid q \geq 1)$ can depict amplitude distribution and change of signal comprehensively and accurately.

III. MULTIFRACTAL ANALYSIS

A. Signal sample

Real-time measuring equipment of L-PLC signal is as Fig. 1. Surveying spot is laboratory building of Hebei University. And the loads are mainly computers and precise instruments. In Fig. 1 the generator emits sine wave of 200kHz with peak-to-peak value 20V. The number of measured data is 2400 each time.
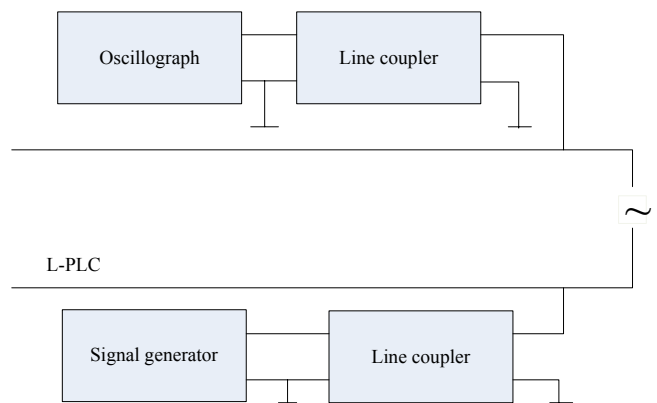

Figure 1. The real-time measuring equipment

B. Calculation & analysis of fractal channel parameter

Real-time measuring data mentioned above are adopted in this paper. We calculate multifractal dimensions of data by using correlation integral algorithm. The algorithm is defined as follows.

At one amplitude point x_j, local density function is defined as

$$p_j(r) = \sum_{k \neq j} \frac{\theta(r - |x_k - x_j|)}{m} \qquad (3)$$

Where m is the number of measured points, $\theta(x)$ is the Heaviside step function defined as

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad (4)$$

With the generalized correlation function $C_q(r)$ given by

$$C_q(r) = \frac{1}{m} \sum_{i=1}^{m} \left[\frac{1}{m} \sum_{j=1}^{m} \theta(r - |x_i - x_j|) \right]^{q-1}$$

$$= \left(\sum_{j=1}^{m} p_j(r)^q \right)^{1/(q-1)} \qquad (5)$$

According to (2), D_q is supposed to satisfy under equation

$$C_q(r) = \left(\sum_{j=1}^{m} p_j(r)^q \right)^{1/(q-1)} \propto r^{D_q} \qquad (6)$$

Based on above equations, D_q is the slope of each point in $C_q(r)$ vs. r log-log plot. And the slope can be calculated by the least squares fitting.

Fig. 2 and Fig. 3 show multifractal dimensions of signal for different cases. Fig. 2 shows multifractal dimensions D_q vs. q for signal with and without load. Fig. 3 shows multifractal dimensions D_q vs. q or signals with different duration.

Fig. 2 reveals that there are great distinct between two curves. It is apparent that dimension with load is larger than without load. That is because original signal becomes more complex after added load. And it causes original amplitude distribution more asymmetrical, so larger dimension is needed to describe the more complex signal.

And then the shapes of these two curves are different. Curve 1 is convex, and curve 2 is concave. That means when q is small, D_q is easily influenced by applied load. In other words, distribution of region with few points is easily influenced by applied load. And distributions of sets with lots of points are more stable.

Fig. 2 and Fig. 3 show that diverse signal propagation environment lead to different distribution in each amplitude set. And variational singularity intensity of amplitude sets will effect change of D_q. This is to say that D_q will be varied as the number of points distributed in each set changes. Therefore, D_q reveals how propagation environments influence the intensity of received signal. In short, different environments show different propagation properties. And transmitted signals in different environments show various distributions of voltage amplitude sets. In this way multifractal

dimensions of L-PLC signal are different in diverse conditions of propagation. And for the same channel propagation environment dimension remains fundamentally unchanged. This is because the same inherent characteristics of attenuation and multipath will cause similar distribution of amplitude sets.
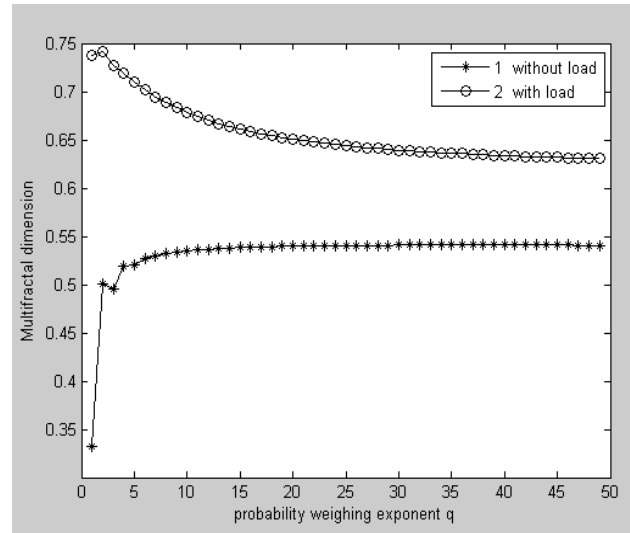

Figure 2.   Multifractal dimensions D_q vs. q for signal with and without load
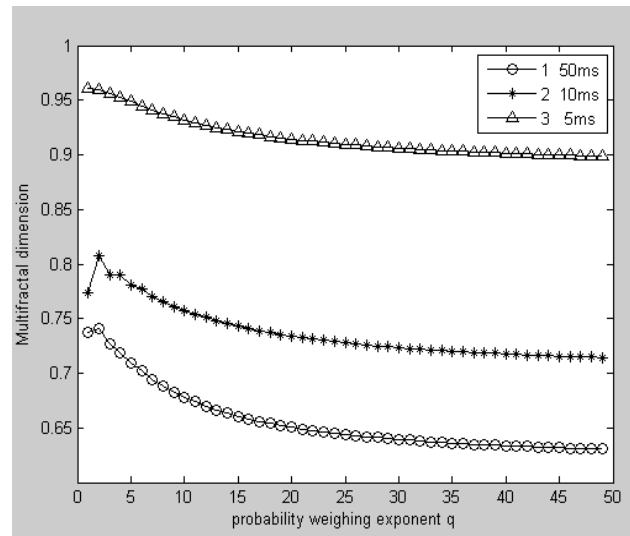

Figure 3.   Multifractal dimensions D_q vs. q for signal of different duration

For above-mentioned reasons multifractal dimensions can be used as primary parameter for describing chaotic channel characteristics and modeling the channel.

IV. FRACTIONAL BROWNIAN MOTION

Generally speaking, subjects mostly have complex details in microscopic size. Due to the limitation of measurement technology, investigated subject only can be observed and studied in comparatively large scale. Methods of linear interpolation and spline fit are often

used to estimate smaller scale data through the data observed in large scale when more detailed characteristics is needed [19-20]. The hypothesis of the method is that small scale details of subject are complex and nonlinear. Therefore, it is not accurate to analyze this kind of subjects inevitably. Besides, sometimes it is impossible to ensure the dimension of some subjects, such as L-PLC signal. Fractal theory describes the comparability of macrostructure and microstructure fitly. If fractal dimension of subject can be calculated accurately, the smaller scale information will be interpolated based on the data of large scale owing to the comparability of whole and part of anomalistic subject. In addition, according to fractal dimension the development trend of subject can be predicted. Therefore, fractal method is able to avoid above-mentioned faults. An interpolation method based on classical FBM is proposed as follows.

FBM is a natural extension of ordinary Brownian motion. It is a Gaussian zero-mean nonstationary stochastic process $B_H(t)$, indexed by the Hurst parameter H in the interval (0, 1). It is a basic mathematical model for fractal process. A fractional Brownian motion $B_H(t)$ is defined as follows.

1) $B_H(0) = 0$, and $B_H(t)$ is continuous.

2) for any $t \geq 0$ and $\Delta t > 0$ increment $\Delta B_H(t) = B(t + \Delta t) - B(t)$ is a Gaussian process with zero-mean and $\delta^2(\Delta t)^{2H}$-variation. It can be expressed as

$$p_r[\frac{B_H(t + \Delta t) - B_H(t)}{|\Delta t|} < t] = F(t) \tag{7}$$

Where $F(t)$ is a distribution function of Gaussian random variable with zero-mean and δ^2-variance. $0 < H < 1$ is fractional calibration parameter, namely Hurst parameter, which determines roughness concentration of FBM.

FBM process can be realized by random midpoint displacement algorithm. The method is as follows.

Define a fractional Brownian motion $B_H(t)$. $B_H(0)$ and $B_H(1)$ are known. Then $B_H(1/2)$ can be constructed by

$$B_H(1/2) = [B_H(0) + B_H(1)]/2 + \Delta_1 \tag{8}$$

Where Δ_1 is a Gaussian random variable with zero-mean and δ_1^2-variance. According to the definition of FBM, $Var[B_H(t_2) - B_H(t_1)]$ is expressed as

$$Var[B_H(t_2) - B_H(t_1)] = \delta^2 |t_2 - t_1|^{2H} \tag{9}$$

Then δ_1^2 can be calculated by

$$\delta_1^2 = Var[B_H(1/2) - B_H(0)] - \frac{1}{4}Var[B_H(1) - B_H(0)]$$

$$= \frac{\delta^2}{2^{2H}}(1 - 2^{2H-2}) \tag{10}$$

Based on (8) and (10), $B_H(1/2)$ can be calculated. Similarly, δ_n^2 can be expressed as

$$\delta_n^2 = \frac{\delta^2}{(2^n)^{2H}}(1 - 2^{2H-2}) \tag{11}$$

In this way subtle structure can be constructed in any scalar. This is to say that constructed structure is subtler as the number of (1) iterated more times.

V. EXPERIMENT RESULTS & DISCCUSIONS

L-PLC chaotic signal can be reconstructed by FBM model with multifractal parameters calculated.

Firstly, multifractal dimension D_q must be calculated.

Secondly, Housdorff dimension D_h will be determined by

$$v_q = (q - 1)D_q \tag{12}$$

Then the Hurst parameter H can be computed by

$$H = 2 - D_h \tag{13}$$

In this way δ_1^2 will be calculated by (10).

Thirdly, increment is supposed to be satisfied statistical self-similarity. In this experiment the number of data points is 2400, and suppose if the data is expressed as $x(n)(n = 1, 2, ..., 2400)$ shown in Fig. 4, then the increment $d(n) = x(n+1) - x(n)$ shown in Fig. 5. And Fig. 6 shows statistical probability distribution of $d(n)$. It can be seen that the statistical probability distribution of $d(n)$ is similar to Gaussian distribution. That means received signal is matched with FBM. Therefore, FBM model can be applied to dealing with received signal data.

Finally, FBM model is established through the multifractal parameters calculated. Fig. 7, Fig. 8, Fig. 9 and Fig. 10 show reconstructed signals with different numbers of interpolated points. N stands for the number of interpolated points in each interpolation.
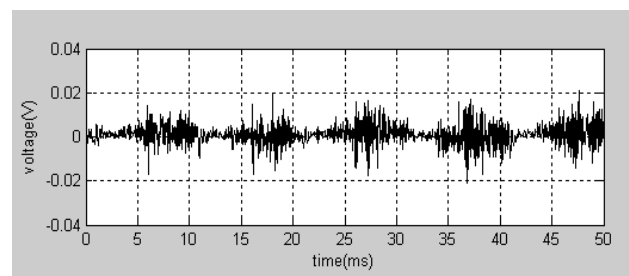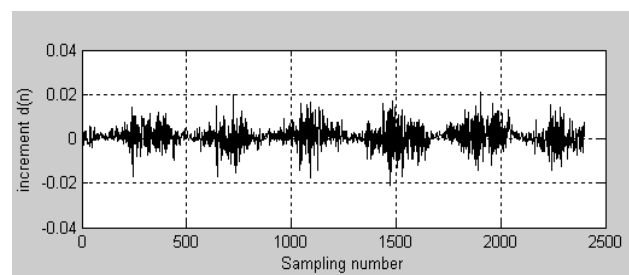

Figure 4.   The original signal

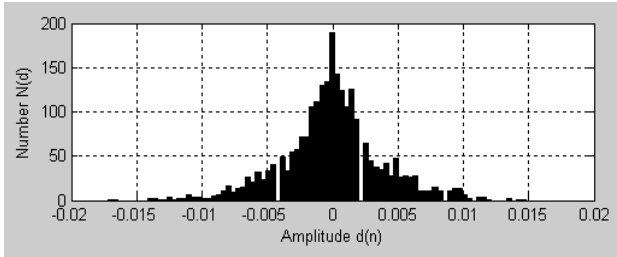Figure 5.   The amplitude of increment d(n)

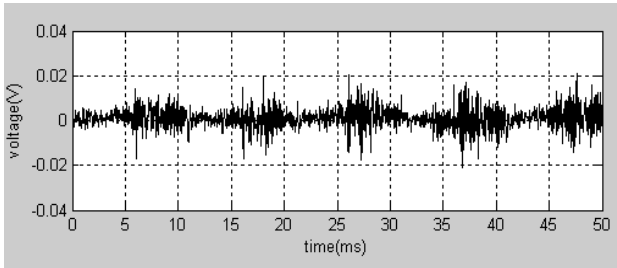Figure 6.   The statistics distribution of increment d(n)

Figure 7.   The reconstructed signal (n=1)
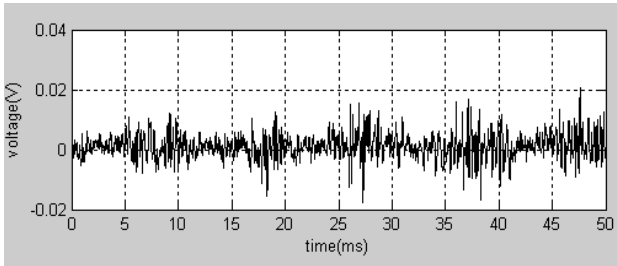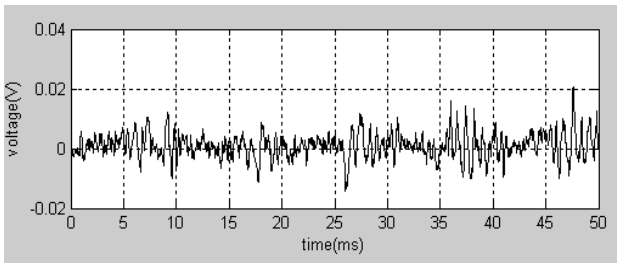

Figure 8.   The reconstructed signal (n=3)

Figure 9.   The reconstructed signal (n=5)
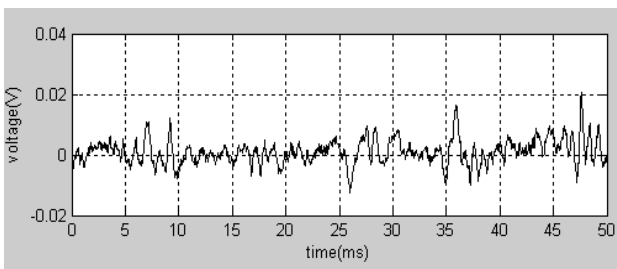

Figure 10. The reconstructed signal (n=7)

Mean square error (MSE) of different reconstructed time series are shown in Fig. 11 which indicates that MSE will rise as the number of interpolation increasing. The reason is that more interpolated points mean less sampled points. Therefore, MSE will increase accordingly. However, when *n* is more than 5, the

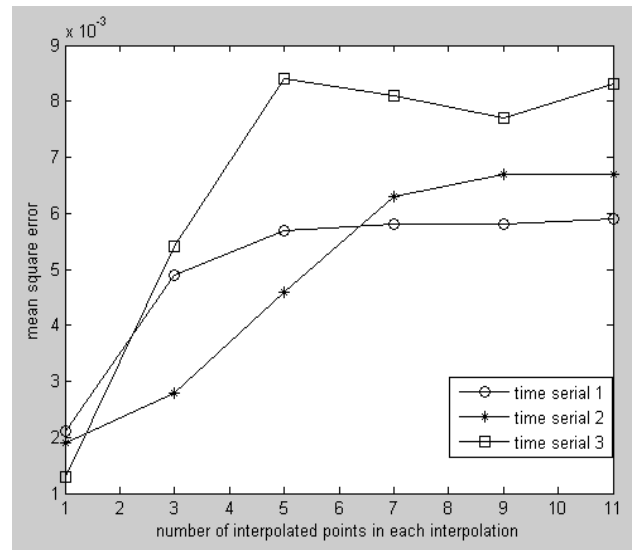change of MSE is not that great. Table1 is the true value of MSE in Fig. 11.

Figure 11.   The MSE of different time series

TABLE I.   THE MSE OF RECONSTRUCTED SIGNALS

Time Series n	1	2	3
1	0.0021	0.0019	0.0013
3	0.0049	0.0028	0.0054
5	0.0057	0.0046	0.0084
7	0.0058	0.0063	0.0081
9	0.0058	0.0067	0.0077
11	0.0059	0.0067	0.0083

VI. CONCLUSION

Due to frequency attenuation, noise interference, multipath interference, etc, communication channels often show chaotic characteristic. For the tight relationship between chaos and fractal, multifractal theory is introduced into the research of chaotic channels in this paper, and a FBM model of chaotic channel is established. We take L-PLC channel, which is a typical chaotic channel, as an example to illustrate the method of modeling chaotic channels. By analyzing increments probability distribution, statistical self-similarity, and multifractal dimensions of L-PLC observed signal, we indicate the inherent relationship between chaotic channel and FBM model, and find that multifractal dimensions can describe the characteristics of chaotic channel effectively and synthetically. Finally, the multifractal model of chaotic channel based on FBM generating algorithm is established with the multifractal parameters calculated and used to reconstruct received signals accurately.

REFERENCES

[1] Fang Shuguang, "Characteristics of low voltage power line communication channels," Relay, vol. 16, no. 22, 2007, pp53-56.

[2] Manfred Zimmermann, Klaus Dostert, "A Multipath Model for the Powerline Channel," IEEE Transactions on Communications, vol.50, 2002, pp553-559.

[3] Wang Zhenchao, Hou Huiran, Gan Yutao, "Research on modeling of the low voltage power line comunication channel based on chaos theory," Electrical Measurement & Instrumentation, vol. 44, no. 500, August 2007, pp20-24.

[4] Wang Zhenchao, Zhang Shibing, Hou Huiran, Gan Yutao, "Analysis of low voltage PLC channel characteristic based on Kolmogorove entropy," Application Research of Computers, vol. 26, no. 4, April 2009, pp. 1225-1229.

[5] Wang Shuxun, He Kai, "The Chaotic Characteristics of Transistor Noise at Low Frequency," IEEE int Conf, Neural Networks and Signal Processing, vol. 1, 2003, pp643-649.

[6] Chen Hongbin, Feng Jiuchao, "Noncoherent CSK communication system and its performance analysis," Journal on Communications, vol. 28, no. 6, June 2007, pp80-84.

[7] Zhu Zhiyu, Jiang Changsheng, "Channel assignment strategy in mobile communication based on chaos neural network," Journal of Electronics & Information Technology, vol. 27, no. 9, September 2005, pp1429-1432.

[8] Zhang Wei, Wu Zhiming, Yang Genke, "Genetic programming modeling on chaotic time series," Acta Electronica Sinica , vol.33, no. 4, April 2005, pp748-751.

[9] Mao Jianqing, Yao Jian, Ding Haishan, "Chaotic time series prediction based on fuzzy tree," Acta Physica Sinica, vol. 58, no. 4, April 2009, pp2220-2230.

[10] Zhai Mingyue, Zheng Qingan, "Markov characterization of LV power line communications systems channels, " Proceedings of the CSEE, vol. 27, no. 22, August 2007, pp116-121.

[11] Hu Gang, Zhu Shihua, Xie Bo, "Analysis of multipath fading channels using chaos and fractal theories," Acta Electronica Sinica, vol. 31, no. 7, July 2003, pp1039-1042.

[12] Heinz-Otto Peitgen, Hartmut Jurgens, Dietmar Saupe, Chaos and fractals, Springer, 2004.

[13] A.I. Olemskoi, V.O. Kharchenko, V.N. Borisyuk, "Multifractal spectrum of phase space related to generalized thermostatistics," Physica A: Statistical Mechanics and its Applications, vol. 387, no. 8, 2008, pp1895-1906.

[14] Zhou Weixing, Didier Sornette, "Numerical investigations of discrete scale invariance in fractals and multifractal measur," Physica A: Statistical Mechanics and its Applications, vol. 388, no. 13, 2009, pp2623-2639.

[15] Antonio Turiel, Conrad J. Perez-Vicente, Jacopo Grazzini, "Numerical methods for the estimation of multifractal singularity spectra on sampled data: A comparative study," Journal of Computational Physics, vol. 216, no. 1, 2006, pp362-390.

[16] De Junfeng, Ka SingLau, "Multifractal formalism for self-similar measures with weak separation condition," Journal de Mathematiques Pures et Appliques, vol. 92, no. 4, 2009, pp407-428.

[17] Yan Guanghui, Li Zhanhuai, Dang Jianwu, "Finding natural cluster hierarchies based on multifractal," Journal of Software, vol. 19, no. 6, June 2008, pp1283-1300.

[18] Long Tujing, Sun Zhengshun, Li Chunwen, Jiang Peigang, Liu Jinhua, "A new multifractal wavelet model for network traffic," Chinese Journal of Computers, vol. 27, no. 8, August 2004, pp1074-1081.

[19] Hu Gang, Zhu Shihua, Xie Bo, "Fractal Brownian motion model of multipath fading channels," Acta Electronica Sinica, vol. 31, no. 1, January 2003, pp137-142.

[20] Zhang Qiang, Xu Jin, "Some models of self similar processes," Journal of China institute of communications, vol. 22, no. 2, 2001, pp106-112.

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