Granular Reasoning and Decision System's Decomposition

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***Abstract*—In order to make further researches on decision logic, a generalized form of decision reasoning called granular reasoning is introduced, which is induced by the inclusion relation over granules, and is connected with granule's operations. The investigations on it shows that granular reasoning not only covers a wider range including decision reasoning, but also satisfies the rules of deduction in classical logic. Thus, granular reasoning can be taken as a foundation to support decision reasoning. On the other hand, granular reasoning has close links with a decision system. Based on granular reasoning, a decision system can be divided into sub-decision systems. Consequently, as a special form of granular reasoning, decision reasoning defined in a decision system becomes to rely on sub-decision systems. This makes decision reasoning easier.**

***Index Terms*—granular reasoning, granule, granular space, decision system, decision reasoning**

I. INTRODUCTION

Decision logic[1] based on a knowledge representation system achieves the goal of reasoning about knowledge. It reflects the idea of integrating reasoning with data information. Although decision logic succeeded in making data reasoning, the researcher did not pay attention to the discussion on the relationship between the reasoning in classical logic and the data reasoning. Actually, decision logic mainly focused discussions on reduction problems, such as reductions of decision rules, reductions of the decision algorithm, reductions of the knowledge representation system itself, etc. This therefore offers us an opportunity to make researches on the connections between classical logic and decision logic. Also, this encourages us to lay a foundation for the data reasoning. In order to do this, we will study data reasoning in a wider range covering decision logic. Thus, we need to review how data reasoning is carried out in decision logic. This will provide support for introducing a generalized form of data reasoning. For the sake of argument, we call a knowledge representation system a decision system, and we refer to the data reasoning in decision logic as decision reasoning.

Consider a decision system $\boldsymbol{S}=(D, A, V, f)$, where D is the universal set; $A=\{c_1,\dots, c_n, d_1,\dots, d_m\}$ is the attribute set, here $c_1,\dots, c_n$ stand for condition attributes, and $d_1,\dots, d_m$ for decision attributes; V, a finite set, is the range of f, each element in V is called a value; and f is the information function from $D\times A$ to V satisfying for each $<z, b>\in D\times A$, there is a value $v\in V$, such that $f(z, b)=v$. Generally, $f(z, b)=v$ is abbreviated to $b(z)=v$. Thus, each attribute $b(\in A)$ is a function from D to V.

Decision reasoning based on the decision system $\boldsymbol{S}=(D, A, V, f)$ is closely connected with formulas which are composed of attributes in A and values in V. For example, when $b\in A$ and $v\in V$, notation (b, v) is called an atomic formula[1]. For $z\in D$, if $b(z)=v$, z is said to satisfy (b, v). Let $|(b, v)|=\{z\,|\,z\in D \text{ and } b(z)=v\}$. Hence, $|(b, v)|$ consists of the data which satisfy (b, v), and $|(b, v)|$ is a subset of D. In this paper, we call $|(b, v)|$ a granule.

Combining atomic formulas by connectives $\neg$, $\wedge$, $\vee$ or $\rightarrow$, compound formulas can be obtained, such as $\neg(b_1, v_1)$, $(b_1, v_1)\wedge(b_2, v_2)$, $(b_1, v_1)\vee(b_2, v_2)$ and $(b_1, v_1)\rightarrow(b_2, v_2)$ are formulas in decision logic, where $b_1, b_2\in A$, and $v_1, v_2\in V$. Applying induction on the length of formulas, we know that if φ and ψ are formulas, then $\neg\varphi$, $\varphi\wedge\psi$, $\varphi\vee\psi$ and $\varphi\rightarrow\psi$ are also formulas. In order to describe decision reasoning, we need to consider such formulas which are of the form:

$$(c_1, v_1)\wedge\dots\wedge(c_n, v_n)\rightarrow(d_1, u_1)\wedge\dots\wedge(d_m, u_m),$$

where $c_1,\dots, c_n$ are all condition attributes, $d_1,\dots, d_m$ are all decision attributes in A, and $v_1,\dots, v_n, u_1,\dots, u_m\in V$. This formula is called a decision rule[1] on $\boldsymbol{S}$.

For this decision rule, using granules $|(c_i, v_i)|$(i=1,…, n) and $|(d_j, u_j)|$ (j=1,…, m), we can obtain $|(c_1, v_1)\wedge\dots\wedge(c_n, v_n)|$ and $|(d_1, u_1)\wedge\dots\wedge(d_m, u_m)|$, each of them is also called a granule in this paper, which are defined as follows:

$$|(c_1, v_1)\wedge\dots\wedge(c_n, v_n)|=|(c_1, v_1)|\cap\dots\cap|(c_n, v_n)|,$$
$$|(d_1, u_1)\wedge\dots\wedge(d_m, u_m)|=|(d_1, u_1)|\cap\dots\cap|(d_m, u_m)|.$$

Since $|(c_i, v_i)|\subseteq D$(i=1,…, n) and $|(d_j, u_j)|\subseteq D$(j=1,…, m),

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we have $|(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)|\subseteq D$ and $|(d_1, u_1)\wedge\ldots\wedge(d_m, u_m)|\subseteq D$. Granules $|(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)|$ and $|(d_1, u_1)\wedge\ldots\wedge(d_m, u_m)|$ play a major role in the definition of decision reasoning. They can decide whether decision reasoning holds or not. Now consider the following definition:

Definition 0[1] Let $\boldsymbol{S}=(D, A, V, f)$ be a decision system, and let $(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)\rightarrow(d_1, u_1)\wedge\ldots\wedge(d_m, u_m)$ be a decision rule on $\boldsymbol{S}$. if $|(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)|\subseteq|(d_1, u_1)\wedge\ldots\wedge(d_m, u_m)|$, then it is to say that this decision rule is true, or to say that $(d_1, u_1)\wedge\ldots\wedge(d_m, u_m)$ can be deduced from $(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)$, denoted by the following expression:

$$(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)\Rightarrow(d_1, u_1)\wedge\ldots\wedge(d_m, u_m). \quad (1)$$

The process of deciding whether (1) holds or not is referred to as decision reasoning. □

Thus, in decision logic, decision reasoning is linked to decision rules, and is connected with the granules $|(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)|$ and $|(d_1, u_1)\wedge\ldots\wedge(d_m, u_m)|$ which consist of data. Therefore, decision reasoning realizes the data reasoning.

The following researches will be made in a wider range including a decision system. The aim is to establish a connection between the reasoning in classical logic and decision reasoning in decision logic. For this purpose, we are going to develop decision logic and to introduce a generalized form of decision reasoning. Fortunately, the developments in [2-12] have provided methods which could be adopted by us. Although there are quite differences among those, they all concern the study of reasoning, data processing, or granules. This naturally offers us research ideas for further investigations.

Because decision reasoning correlates with granules, such as $|(c_1, v_1)\wedge\ldots\wedge(c_n, v_n)|$ and $|(d_1, u_1)\wedge\ldots\wedge(d_m, u_m)|$, the generalized form of decision reasoning to be introduced must correlate with granules as well. So it is required to make it clear what is a granule. In fact, a series of concepts, such as formulas, granular space, granules, granular computing, and so on, have been defined in [2]. These concepts are preparations for the following study. Now, let us sketch them out.

II. Granular Spaces and Granules

The concepts of a granular space and a granule have been defined in [2]. The following definitions 1 to 5 are merely a review for them. The aim is to consider the overall soundness of our technique.

Let U be a finite set called a universal set. Let U^n ($n\geqslant 1$) stand for the Cartesian product $U\times\cdots\times U$ of n factors of U. Each element in U^n is denoted by $<a_1,\ldots, a_n>$, and is called an n-tuple. When $H\subseteq U^n$, H is referred to as an n-place relation on U. If $<a_1,\ldots, a_n>\in H$, the elements $a_1,\ldots, a_n$ have the property described by H. When n=1, the n-tuple $<a_1,\ldots, a_n>$ is $<a_1>$ which is generally written as a_1. Although n-place relations can express various actual problems, we need to consider combinations of relations because actual problems are complicated in many situations. As a formal language, logical formulas are very effective for expressing complicated concepts. So, we will construct formulas by applying n-place relations. According to the general steps in mathematical logic, it is required to introduce a symbolic system.

Definition 1[2]. Let U be a universal set. The symbolic system on U consists of constants, variables, terms, relations and connectives which are given as follows:

1) Constant: for $a\in U$, a is taken as a constant on U.

2) Variable: $x_1, x_2, x_3, \ldots$ are used to denote variables on U.

3) Term: constants and variables are called terms on U, and t_i (i=1, 2,…) is used to stand for any term.

4) Relation: P, Q, S, H, R etc. or $P_1, P_2, P_3,\ldots$ denote n-place relations on U ($n\geqslant 1$).

5) Connective: $\neg, \wedge, \vee, \rightarrow$. □

In mathematical logic, an n-place relation P in this definition is called a predicate interpreted as a relation. Because this paper only concerns the relations on U, an n-place relation P can be viewed as the predicate which is interpreted as P itself.

Using this symbolic system, atomic formulas can be defined, and formulas will be based on atomic formulas.

Definition 2[2]. Let U be a universal set, and let P ($\subseteq U^n$) be an n-place relation. If $t_1,\ldots, t_n$ are n terms, then $P(t_1,\ldots, t_n)$ is called an atomic formula on U. □

Definition 3[2]. Let U be a universal set, then the formulas on U are inductively defined as follows:

1) Every atomic formula on U is a formula on U.

2) If φ is a formula on U, then $\neg\varphi$ is a formula on U.

3) If φ and ψ are formulas on U, then $\varphi\wedge\psi$, $\varphi\vee\psi$ and $\varphi\rightarrow\psi$ are formulas on U.

4) Formulas are generated by using 1), 2) or 3) in finite steps. □

The formulas defined here cover the formulas in decision logic. We will make an explanation on this in a moment.

Now, let Form(U) denote the set of all formulas on U. If $\varphi\in$Form(U) and there exist n variables $x_1,\ldots, x_n$ in φ, then φ is said to be an n-place formula($n\geqslant 1$). In order to stress the n variables, φ is sometimes denoted by $\varphi(x_1,\ldots, x_n)$. For instance, consider an m-place relation $P(\subseteq U^m)$, if $x_1,\ldots, x_n, a_1,\ldots, a_{m-n}$ are m terms on U, where $x_1,\ldots, x_n$ are variables, $a_1,\ldots, a_{m-n}$ are constants and $1\leqslant n\leqslant m$, then $P(x_1,\ldots, x_n, a_1,\ldots, a_{m-n})$ is an n-place formula. It is actually an atomic formula, and can be written as $\varphi(x_1,\ldots, x_n)$.

Definition 4[2]. Let $\varphi\in$Form(U), and φ be an n-place formula($n\geqslant 1$). Notation $|\varphi|$ is used to represent a subset of U^n, it is inductively defined as follows:

1) If $\varphi=P(x_1,\ldots, x_n, a_1,\ldots, a_{m-n})$($n\leqslant m$) is an atomic formula, where $P\subseteq U^m$, then define $|\varphi|=|P(x_1,\ldots, x_n, a_1,\ldots, a_{m-n})|=\{<t_1,\ldots, t_n>\,|\,<t_1,\ldots, t_n>\in U^n$ and $<t_1,\ldots, t_n, a_1,\ldots, a_{m-n}>\in P\}$. In this case $|\varphi|\subseteq U^n$.

2) If $\varphi=\neg\varphi_1$, and $|\varphi_1|\subseteq U^n$, then define $|\varphi|=|\neg\varphi_1|=\sim|\varphi_1|=U^n-|\varphi_1|$.

3) If $\varphi=\varphi_1\wedge\varphi_2$, and $|\varphi_1|, |\varphi_2|\subseteq U^n$, then define $|\varphi|=|\varphi_1$

$\wedge \varphi_2| = |\varphi_1| \cap |\varphi_2|$.

4) If $\varphi=\varphi_1 \vee \varphi_2$, and $|\varphi_1|$, $|\varphi_2| \subseteq U^n$, then define $|\varphi| = |\varphi_1 \vee \varphi_2| = |\varphi_1| \cup |\varphi_2|$.

5) If $\varphi=\varphi_1 \rightarrow \varphi_2$, and $|\varphi_1|$, $|\varphi_2| \subseteq U^n$, then define $|\varphi| = |\varphi_1 \rightarrow \varphi_2| = |\neg\varphi_1| \cup |\varphi_2|$.

6) In particular, for n-place formula $\varphi \in \text{Form}(U)$ and n=0(such as $\varphi=P(a_1,\ldots, a_m)$, where $a_1,\ldots, a_m$ are all constants), then define $|\varphi|=\varnothing$. In this case, $|\varphi| \subseteq U^m$ for any m$\geqslant$0. □

This definition shows that for each n-place formula $\varphi \in \text{Form}(U)$, $|\varphi|$ is a subset of U^n, i.e. $|\varphi| \subseteq U^n$. When $<t_1,\ldots, t_n> \in |\varphi|$, we say that n-tuple $<t_1,\ldots, t_n>$ satisfies φ. So, $|\varphi|$ consists of the n-tuples satisfying φ.

We have mentioned above that the formulas in definition 3 cover the formulas in decision logic. Now we return to consideration of this fact. Let $\mathbf{S}=(D, A, V, f)$ be a decision system. Consider an atomic formula (b, v) in decision logic, where $b \in A$ and $v \in V$(see section 1). Formula (b, v) can be regarded as an atomic formula in definition 2. In fact, since attribute b is a function from D to V, defined by $b(z)=f(b, z)$ where $z \in D$, and $f(b, z) \in V$, if let $f(b, z)=v(\in V)$ and $U=D \cup V$, then b can be viewed as a binary relation on U, i.e. $b \subseteq U \times U$, such that for $<z, v> \in U \times U$, $<z, v> \in b$ if and only if $b(z)=v$. In this case, the atomic formula (b, v) can be denoted by $b(x_1, v)$, where x_1 is a variable, and v is taken as a constant, they are all terms in definition 1. Moreover, for $z \in U$, z satisfies (b, v), if and only $b(z)=v$(see section 1), if and only if $<z, v> \in b$, if and only if z satisfies $b(x_1\ v)$(see definition 4). This means $|(b, v)|=|b(x_1\ v)|$. Thus (b, v) can be viewed as formula $b(x_1\ v)$ which is an atomic formula in definition 2. Because the formulas in definition 3 are based on n-place relations which contain binary relations, the formulas in definition 3 cover the formulas in decision logic, or the formulas in decision logic are only a part of the formulas in definition 3.

Definition 5[2]. Let U be a universal set, and Form(U) be the formula set on U. The structure consisting of U together with Form(U), denoted by $<U, \text{Form}(U)>$, is called the granular space on U. For each n-place formula $\varphi \in \text{Form}(U)$, φ is also referred to as a formula in $<U, \text{Form}(U)>$, and the set $|\varphi|$, a subset of U^n, is called a granule corresponding to φ. □

Ref. [2] shows that granular space $<U, \text{Form}(U)>$ is an extension of the decision system $\mathbf{S}=(D, A, V, f)$, where $U=D \cup V$. Besides, notice that the formula set of decision logic can be viewed as a part of Form(U). So, granular space $<U, \text{Form}(U)>$ revolves a wider range containing the decision system.

By an informal explanation, a granule is regarded as a part of a whole[8]. Definition 4 shows that $|\varphi|$ is a subset of U^n. If U^n is taken as a whole, then $|\varphi|$ is a part of the whole. This means that $|\varphi|$ defined as a granule coincides with the ordinary understanding.

At first glance, $<U, \text{Form}(U)>$ would not seem to involve any granules at all. However, for any formula $\varphi \in \text{Form}(U)$, φ produces the granule $|\varphi|$. Thus, granules have close links with the formulas in $<U, \text{Form}(U)>$. Since these formulas are connected with granules and cover the formulas of decision logic, it is possible to define a generalized form of decision reasoning which will operate among the formulas in $<U, \text{Form}(U)>$, and will be determined by granules.

III. Granular Reasoning

The operations of granules in definition 4, such as $|\varphi_1| \cap |\varphi_2|=|\varphi_1 \wedge \varphi_2|$, $|\varphi_1| \cup |\varphi_2|=|\varphi_1 \vee \varphi_2|$ etc., are one kind of granular computing which is a current research topic in computer science, and was defined in [2]. So, definition 4 gives properties about granular computing. The reasoning to be defined in this paper will involve these properties, also, it will be linked to the inclusion relation $\subseteq$ over granules, which are all thought of as granular computing in this paper.

A. Definition of Granular Reasoning

We now define a generalized form of decision reasoning which is connected with the formulas in $<U, \text{Form}(U)>$, and concerns granules and granular computing. Consider the following definition.

Definition 6. Let $<U, \text{Form}(U)>$ be the granular space on U, $\Sigma \subseteq \text{Form}(U)$ and $\varphi \in \text{Form}(U)$. If there exist finite formulas $\psi_1,\ldots, \psi_n \in \Sigma$, such that $|\psi_1 \wedge \ldots \wedge \psi_n| \subseteq |\varphi|$, then it is to say that φ is granularly deduced from Σ, denoted by $\Sigma \models \varphi$, and the process of deciding whether $\Sigma \models \varphi$ holds or not is referred to as granular reasoning. □

$\Sigma \models \varphi$, granular reasoning, establishes a connection between finite formulas $\psi_1,\cdots, \psi_n$ in Σ and formula φ. This connection is determined by granules and granular computing, i.e. it depends on whether granule $|\psi_1 \wedge \cdots \wedge \psi_n|$ is contained in granule $|\varphi|$.

If $\Sigma \models \varphi$, then there exist finite formulas $\psi_1,\ldots, \psi_n \in \Sigma$, such that $|\psi_1 \wedge \ldots \wedge \psi_n| \subseteq |\varphi|$. When $\Sigma=\{\gamma_1,\ldots, \gamma_m\}$ is a finite formula set, $\Sigma \models \varphi$ is usually written as $\gamma_1,\ldots, \gamma_m \models \varphi$. In this case, we have the following property:

$\gamma_1,\ldots, \gamma_m \models \varphi$ if and only if $|\gamma_1 \wedge \ldots \wedge \gamma_m| \subseteq |\varphi|$, if and only if $\gamma_1 \wedge \ldots \wedge \gamma_m \models \varphi$.

As the analysis between definition 4 and definition 5, the formulas in decision logic based on the decision system $\mathbf{S}=(D, A, V, f)$ can be viewed as the formulas in Form(U), where $U=D \cup V$. Thus, decision reasoning induced by the inclusion relation $\subseteq$ over granules(see definition 0) is a special form of granular reasoning because according to definition 6, expression (1) can be expressed in terms of granular reasoning:

$$(c_1, v_1),\ldots, (c_n, v_n) \models (d_1, u_1) \wedge \ldots \wedge (d_m, u_m).$$

Also, it can be characterized as follows:

$$(c_1, v_1) \wedge \ldots \wedge (c_n, v_n) \models (d_1, u_1) \wedge \ldots \wedge (d_m, u_m).$$

This representation is the form of granular reasoning. Hence, granular reasoning is a generalized form of decision reasoning.

In section 5, we will apply granular reasoning to separate a decision system into sub-decision systems. This will show an application of granular reasoning.

B. Rules of Deduction

Granular reasoning is determined by the inclusion relation $\subseteq$ over granules. It is very different from formal reasoning in classical logic. We know that formal reasoning based on the natural deduction system[8] in classical logic is produced by rules of deduction which form the natural deduction system. How is the relationship between granular reasoning and the rules of deduction? We now investigate this problem. Let us review the rules in the natural deduction system. Suppose that Σ or Σ' stands for a formula set, and φ, ψ and β are formulas. For the sake of simplicity, we adopt the notation "Σ, φ" to denote set $\Sigma \cup \{\varphi\}$, and "Σ, Σ'" to denote set $\Sigma \cup \Sigma'$. The rules of deduction in classical logic are listed as follows[8]:

($\in$) $\Sigma, \varphi \vdash \varphi$.

($+$) If $\Sigma \vdash \varphi$, then $\Sigma, \Sigma' \vdash \varphi$.

($\neg-$) If $\Sigma, \neg\varphi \vdash \psi$ and $\Sigma, \neg\varphi \vdash \neg\psi$, then $\Sigma \vdash \varphi$.

($\rightarrow+$) If $\Sigma, \psi \vdash \varphi$, then $\Sigma \vdash \psi \rightarrow \varphi$.

($\rightarrow-$) If $\Sigma \vdash \psi$ and $\Sigma \vdash \psi \rightarrow \varphi$, then $\Sigma \vdash \varphi$.

($\wedge+$) If $\Sigma \vdash \psi$ and $\Sigma \vdash \varphi$, then $\Sigma \vdash \psi \wedge \varphi$.

($\wedge-$) If $\Sigma \vdash \psi \wedge \varphi$, then $\Sigma \vdash \psi$ and $\Sigma \vdash \varphi$.

($\vee+$) If $\Sigma \vdash \psi$, then $\Sigma \vdash \psi \vee \varphi$ and $\Sigma \vdash \varphi \vee \psi$.

($\vee-$) If $\Sigma, \psi \vdash \beta$ and $\Sigma, \varphi \vdash \beta$, then $\Sigma, \psi \vee \varphi \vdash \beta$.

In classical logic, the system consisting of these rules is called the natural deduction system[8]. For any formula set Σ and a formula φ, expression $\Sigma \vdash \varphi$ can be described as follows:

There is a sequence $\Sigma_1 \vdash \varphi_1, \ldots, \Sigma_n \vdash \varphi_n (n \geqslant 1)$, where $\Sigma_i (i=1, \ldots, n)$ is a formula set, $\varphi_i (i=1, \ldots, n)$ is a formula, and $\Sigma_i \vdash \varphi_i (i=1, \cdots, n)$ is obtained by using one of the rules in the natural deduction system, satisfying $\Sigma_n = \Sigma$ and $\varphi_n = \varphi$. The process of getting sequence $\Sigma_1 \vdash \varphi_1, \ldots, \Sigma_n \vdash \varphi_n$ is referred to as formal reasoning[8], denoted by $\Sigma \vdash \varphi$.

Thus $\Sigma \vdash \varphi$, formal reasoning, stands for a process of getting sequence $\Sigma_1 \vdash \varphi_1, \ldots, \Sigma_n \vdash \varphi_n$. Note that "$\Sigma_1 \vdash \varphi_1$" in this sequence must be obtained by using rule ($\in$), that is $\varphi_1 \in \Sigma_1$. Formal reasoning is produced by the rules in the natural deduction system, also, it is relevant to formula's form.

Comparing granular reasoning $\Sigma \models \varphi$ with formal reasoning $\Sigma \vdash \varphi$, we notice that one is determined by the inclusion relation $\subseteq$ over granules, and the other is generated by the rules of deduction in the natural deduction system. Their deduction processes are very different.

However, granular reasoning has connections with formal reasoning. Let us examine the rules in the natural deduction system. Each of them is an abstraction for a deduction pattern occurring in some disciplines, such as mathematics, physics, computer science, etc. So formal reasoning actually imitates the deduction processes appearing in the disciplines. If $\Sigma \vdash \varphi$ implies $\Sigma \models \varphi$, then granular reasoning will keep the deduction processes. The key to deriving $\Sigma \models \varphi$ from $\Sigma \vdash \varphi$ depends on whether symbol "$\vdash$" in rules ($\in$), ($+$), ($\neg-$), ($\rightarrow+$), ($\rightarrow-$), ($\wedge+$), ($\wedge-$), ($\vee+$) and ($\vee-$) can be replaced by symbol "$\models$". The following will study this problem.

C. Granular Patterns of Reasoning

If symbol "$\vdash$" in rules ($\in$), ($+$), ($\neg-$), ($\rightarrow+$), ($\rightarrow-$), ($\wedge+$), ($\wedge-$), ($\vee+$) and ($\vee-$) is replaced by symbol "$\models$", some changes will take place in each rule, such as "$\Sigma, \varphi \vdash \varphi$" will become "$\Sigma, \varphi \models \varphi$", and "If $\Sigma \vdash \varphi$, then $\Sigma, \Sigma' \vdash \varphi$" will become "If $\Sigma \models \varphi$, then $\Sigma, \Sigma' \models \varphi$", etc.

Since symbol "$\models$" represents granular reasoning, we call each of the changed rules a granular pattern of reasoning. For example, "If $\Sigma \models \varphi$, then $\Sigma, \Sigma' \models \varphi$" corresponding to ($+$) is a granular pattern of reasoning. Now, the question is, does each of the granular patterns of reasoning describe a correct conclusion? The following theorems will give the answers.

Theorem 1. Let $<U, \mathrm{Form}(U)>$ be the granular space on U, $\Sigma \subseteq \mathrm{Form}(U)$, and $\varphi, \psi \in \mathrm{Form}(U)$. The following granular patterns of reasoning hold:

1) $\Sigma, \varphi \models \varphi$.

2) If $\Sigma \models \varphi$, then $\Sigma, \Sigma' \models \varphi$.

3) If $\Sigma, \neg\varphi \models \psi$ and $\Sigma, \neg\varphi \models \neg\psi$, then $\Sigma \models \varphi$.

Proof 1) Since $\varphi \in \Sigma \cup \{\varphi\}$ and $|\varphi| \subseteq |\varphi|$, it follows from definition 6 that $\Sigma, \varphi \models \varphi$.

2) Suppose that $\Sigma \models \varphi$. Then there are finite formulas $\psi_1, \ldots, \psi_n \in \Sigma$, such that $|\psi_1 \wedge \ldots \wedge \psi_n| \subseteq |\varphi|$. Obviously, $\psi_1, \ldots, \psi_n \in \Sigma \cup \Sigma'$. Thus $\Sigma, \Sigma' \models \varphi$.

3) Suppose that $\Sigma, \neg\varphi \models \psi$ and $\Sigma, \neg\varphi \models \neg\psi$.

By $\Sigma, \neg\varphi \models \psi$, there are finite formulas $\beta_1, \ldots, \beta_n, \neg\varphi \in \Sigma \cup \{\neg\varphi\}$, such that $|\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\varphi| \subseteq |\psi|$. If there exist formulas $\beta_1, \ldots, \beta_n \in \Sigma$, such that $|\beta_1 \wedge \ldots \wedge \beta_n| \subseteq |\psi|$, then $|\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\varphi| = |\beta_1 \wedge \ldots \wedge \beta_n| \cap |\neg\varphi| \subseteq |\beta_1 \wedge \cdots \wedge \beta_n| \subseteq |\psi|$, we still have $|\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\varphi| \subseteq |\psi|$.

By $\Sigma, \neg\varphi \models \neg\psi$, there exist finite formulas $\gamma_1, \ldots, \gamma_m, \neg\varphi \in \Sigma \cup \{\neg\varphi\}$, such that $|\gamma_1 \wedge \ldots \wedge \gamma_m \wedge \neg\varphi| \subseteq |\neg\psi|$, where $\gamma_1, \ldots, \gamma_m \in \Sigma$.

From these results we have $|\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\varphi| \cap |\gamma_1 \wedge \ldots \wedge \gamma_m \wedge \neg\varphi| \subseteq |\psi| \cap |\neg\psi|$. By definition 4, we get $|\beta_1 \wedge \ldots \wedge \beta_n \wedge \gamma_1 \wedge \ldots \wedge \gamma_m| \cap \sim|\varphi| \subseteq \varnothing$. Hence $|\beta_1 \wedge \ldots \wedge \beta_n \wedge \gamma_1 \wedge \ldots \wedge \gamma_m| \subseteq |\varphi|$. Since $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m$ are finite formulas in Σ, we have $\Sigma \models \varphi$. □

Theorem 2. Let $<U, \mathrm{Form}(U)>$ be the granular space on U, $\Sigma \subseteq \mathrm{Form}(U)$, and $\varphi, \psi \in \mathrm{Form}(U)$. The following granular patterns of reasoning hold:

1) If $\Sigma, \psi \models \varphi$, then $\Sigma \models \psi \rightarrow \varphi$.

2) If $\Sigma \models \psi$ and $\Sigma \models \psi \rightarrow \varphi$, then $\Sigma \models \varphi$.

Proof 1) By definition 4, we have:

$$|\psi \rightarrow \varphi| = |\neg\psi| \cup |\varphi| = \sim|\psi| \cup |\varphi|. \quad (2)$$

Suppose that $\Sigma, \psi \models \varphi$. Then there must be finite formulas $\beta_1, \ldots, \beta_n, \psi \in \Sigma \cup \{\psi\}$, such that

$$|\beta_1 \wedge \ldots \wedge \beta_n \wedge \psi| \subseteq |\varphi|, \quad (3)$$

where $\beta_1, \ldots, \beta_n \in \Sigma$.

Since $|\beta_1 \wedge \ldots \wedge \beta_n \wedge \psi| = |\beta_1 \wedge \ldots \wedge \beta_n| \cap |\psi|$, by (3) we have $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq\sim|\psi|\cup|\varphi|$. It follows from (2) that $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi\to\varphi|$. Because $\beta_1,\ldots,\beta_n$ are finite formulas in Σ, we get $\Sigma\models\psi\to\varphi$.

2) Suppose that $\Sigma\models\psi$ and $\Sigma\models\psi\to\varphi$. Then there must be finite formulas $\beta_1,\ldots,\beta_n\in\Sigma$, as well as $\gamma_1,\ldots,\gamma_m\in\Sigma$, such that $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi|$, and $|\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\psi\to\varphi|$ $(=\sim|\psi|\cup|\varphi|)$.

From these results, we can prove $|\beta_1\wedge\ldots\wedge\beta_n\wedge\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\varphi|$. In fact, for any $<t_1,\cdots,t_r>\in|\beta_1\wedge\ldots\wedge\beta_n\wedge\gamma_1\wedge\ldots\wedge\gamma_m|$ $(r\geqslant1)$, since $|\beta_1\wedge\ldots\wedge\beta_n\wedge\gamma_1\wedge\ldots\wedge\gamma_m|=|\beta_1\wedge\ldots\wedge\beta_n|\cap|\gamma_1\wedge\ldots\wedge\gamma_m|$, we have $<t_1,\ldots,t_r>\in|\beta_1\wedge\ldots\wedge\beta_n|$ and $<t_1,\ldots,t_r>\in|\gamma_1\wedge\ldots\wedge\gamma_m|$. So, $<t_1,\ldots,t_r>\in|\psi|$, and $<t_1,\ldots,t_r>\in\sim|\psi|\cup|\varphi|$. These means $<t_1,\ldots,t_r>\in|\varphi|$. Thus $|\beta_1\wedge\ldots\wedge\beta_n\wedge\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\varphi|$. Since $\beta_1,\ldots,\beta_n$, $\gamma_1,\ldots,\gamma_m$ are finite formulas in Σ, we have $\Sigma\models\varphi$. □

Theorem 3. Let $<U$, Form$(U)>$ be the granular space on U, $\Sigma\subseteq$Form(U), and $\varphi,\psi\in$Form(U). The following granular patterns of reasoning hold:

1) If $\Sigma\models\psi$ and $\Sigma\models\varphi$, then $\Sigma\models\psi\wedge\varphi$.

2) If $\Sigma\models\psi\wedge\varphi$, then $\Sigma\models\psi$ and $\Sigma\models\varphi$.

Proof 1) Suppose that $\Sigma\models\psi$ and $\Sigma\models\varphi$. Then there must be formulas $\beta_1,\ldots,\beta_n\in\Sigma$, and $\gamma_1,\ldots,\gamma_m\in\Sigma$, such that $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi|$, and $|\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\varphi|$. Since $|\beta_1\wedge\ldots\wedge\beta_n\wedge\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi|$, and $|\beta_1\wedge\ldots\wedge\beta_n\wedge\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\varphi|$, we have $|\beta_1\wedge\ldots\wedge\beta_n\wedge\gamma_1\wedge\ldots\wedge\gamma_m|\subseteq|\psi|\cap|\varphi|=|\psi\wedge\varphi|$. Because $\beta_1,\ldots,\beta_n$, $\gamma_1,\ldots,\gamma_m$ are finite formulas in Σ, this means $\Sigma\models\psi\wedge\varphi$.

2) Suppose that $\Sigma\models\psi\wedge\varphi$. Then there exist finite formulas $\beta_1,\ldots,\beta_n\in\Sigma$, such that $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi\wedge\varphi|$. Because $|\psi\wedge\varphi|=|\psi|\cap|\varphi|\subseteq|\psi|$, and $|\psi\wedge\varphi|=|\psi|\cap|\varphi|\subseteq|\varphi|$, we have $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi|$, and $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\varphi|$. Hence, $\Sigma\models\psi$ and $\Sigma\models\varphi$. □

From this theorem, we immediately get the following corollary:

Corollary 1 Let $<U$, Form$(U)>$ be the granular space on U, $\Sigma\subseteq$Form(U), and $\varphi,\psi\in$Form(U). Then:

$\Sigma\models\psi\wedge\varphi$ if and only if $\Sigma\models\psi$ and $\Sigma\models\varphi$. □

Theorem 4. Let $<U$, Form$(U)>$ be the granular space on U, $\Sigma\subseteq$Form(U), and $\varphi,\psi,\beta\in$ Form(U). The following granular patterns of reasoning hold:

1) If $\Sigma\models\psi$, then $\Sigma\models\psi\vee\varphi$ and $\Sigma\models\varphi\vee\psi$.

2) If $\Sigma,\psi\models\beta$ and $\Sigma,\varphi\models\beta$, then $\Sigma,\psi\vee\varphi\models\beta$.

Proof 1) Suppose that $\Sigma\models\psi$. Then there must be finite formulas $\beta_1,\ldots,\beta_n\in\Sigma$, such that $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi|$. Because $|\psi|\subseteq|\psi|\cup|\varphi|=|\psi\vee\varphi|$ and $|\psi|\subseteq|\varphi|\cup|\psi|=|\varphi\vee\psi|$, we get $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\psi\vee\varphi|$ and $|\beta_1\wedge\ldots\wedge\beta_n|\subseteq|\varphi\vee\psi|$. Thus, $\Sigma\models\psi\vee\varphi$ and $\Sigma\models\varphi\vee\psi$.

2) Suppose that $\Sigma,\psi\models\beta$ and $\Sigma,\varphi\models\beta$. Then there must be formulas $\gamma_1,\ldots,\gamma_n,\psi\in\Sigma\cup\{\psi\}$, and $\tau_1,\ldots,\tau_m,\varphi\in\Sigma\cup\{\varphi\}$, such that $|\gamma_1\wedge\ldots\wedge\gamma_n\wedge\psi|\subseteq|\beta|$, and $|\tau_1\wedge\ldots\wedge\tau_m\wedge\varphi|\subseteq|\beta|$, where $\gamma_1,\ldots,\gamma_n\in\Sigma$ and $\tau_1,\ldots,\tau_m\in\Sigma$. Thus, the following result is true:

$$|\gamma_1\wedge\ldots\wedge\gamma_n\wedge\psi|\cup|\tau_1\wedge\ldots\wedge\tau_m\wedge\varphi|\subseteq|\beta|.$$

Because $|\gamma_1\wedge\ldots\wedge\gamma_n\wedge\tau_1\wedge\ldots\wedge\tau_m\wedge(\psi\vee\varphi)|=|(\gamma_1\wedge\ldots\wedge\gamma_n\wedge\tau_1\wedge\ldots\wedge\tau_m\wedge\psi)\vee(\gamma_1\wedge\ldots\wedge\gamma_n\wedge\tau_1\wedge\ldots\wedge\tau_m\wedge\varphi)|\subseteq|\gamma_1\wedge\ldots\wedge\gamma_n\wedge\psi|\cup|\tau_1\wedge\ldots\wedge\tau_m\wedge\varphi|$, we get $|\gamma_1\wedge\ldots\wedge\gamma_n\wedge\tau_1\wedge\ldots\wedge\tau_m\wedge(\psi\vee\varphi)|\subseteq|\beta|$. Since $\gamma_1,\ldots,\gamma_n,\tau_1,\ldots,\tau_m,\psi\vee\varphi$ are finite formulas in $\Sigma\cup\{\psi\vee\varphi\}$, this deduce the conclusion: $\Sigma,\psi\vee\varphi\models\beta$. □

From theorems 1 to 4, we know that if symbol "$\vdash$" in rules $(\in)$, $(+)$, $(\neg-)$, $(\to+)$, $(\to-)$, $(\wedge+)$, $(\wedge-)$, $(\vee+)$ and $(\vee-)$ is replaced by symbol "$\models$", the granular patterns of reasoning are correct conclusions. So, these patterns can be taken as rules for granular reasoning. Using these conclusions, we can discuss the connections between $\Sigma\models\varphi$ and $\Sigma\vdash\varphi$.

IV. SOUNDNESS AND COMPLETENESS

It is well known that classical logic is the basis of other logics. As a major aspect of discussion, a systematic study of formal reasoning has been made in classical logic. Because formal reasoning is based on the rules of deduction in the natural deduction system, and the granular patterns of reasoning being correct conclusions correspond to these rules, it is ready for us to examine the relationship between $\Sigma\models\varphi$ and $\Sigma\vdash\varphi$.

In classical logic, the natural deduction system, consisting of rules $(\in)$, $(+)$, $(\neg-)$, $(\to+)$, $(\to-)$, $(\wedge+)$, $(\wedge-)$, $(\vee+)$ and $(\vee-)$, is equivalent to a related formal system of axioms. Because these rules are abstractions of deduction patterns occurring in some disciplines, they are easily accepted. When considering these rules on the granular space $<U$, Form$(U)>$, we obtain the natural deduction system of space $<U$, Form$(U)>$, which still consists of rules $(\in)$, $(+)$, $(\neg-)$, $(\to+)$, $(\to-)$, $(\wedge+)$, $(\wedge-)$, $(\vee+)$ and $(\vee-)$. Thus, we can conduct formal reasoning in the granular space. More specifically, for $\Sigma\subseteq$Form(U) and $\varphi\in$Form(U), being similar to classical logic, the notation "$\Sigma\vdash\varphi$" is still applied to denote formal reasoning in space $<U$, Form$(U)>$, which is described as follows:

If there exist formula sets $\Sigma_1,\ldots,\Sigma_n\subseteq$Form$(U)$, and formulas $\varphi_1,\ldots,\varphi_n\in$ Form(U), such that there is a sequence $\Sigma_1\vdash\varphi_1,\ldots,\Sigma_n\vdash\varphi_n$, where each $\Sigma_i\vdash\varphi_i$ (i=1,…, n) is obtained by using one of the rules in the natural deduction system, and $\Sigma_n=\Sigma$, as well as $\varphi_n=\varphi$, then the process of getting sequence $\Sigma_1\vdash\varphi_1,\ldots,\Sigma_n\vdash\varphi_n$ is referred to as formal reasoning, denoted by $\Sigma\vdash\varphi$.

Thus, formal reasoning, i.e. sequence $\Sigma_1\vdash\varphi_1,\ldots,\Sigma_n\vdash\varphi_n$, is generated by the rules of deduction, and the first expression $\Sigma_1\vdash\varphi_1$ must be obtained by rule $(\in)$. Also, formal reasoning depends on formula's form.

A. Soundness on Granular Space

Let $<U$, Form$(U)>$ be the granular space on U, $\Sigma\subseteq$Form(U), and $\varphi\in$Form(U). If $\Sigma\vdash\varphi$ implies $\Sigma\models\varphi$, then the natural deduction system is said to have the property of soundness. Notice that the soundness is connected with the granular space $<U$, Form$(U)>$, we also call it the soundness on granular space.

Since the granular patterns of reasoning are correct conclusions, and each $\Sigma_i \vdash \varphi_i$ (i=1,…, n) in sequence $\Sigma_1 \vdash \varphi_1, \ldots, \Sigma_n \vdash \varphi_n$ is obtained by using one of the rules in the natural deduction system, this means that sequence $\Sigma_1 \vdash \varphi_1, \ldots, \Sigma_n \vdash \varphi_n$ can be changed into $\Sigma_1 \models \varphi_1, \ldots, \Sigma_n \models \varphi_n$. Because $\Sigma_n = \Sigma$ and $\varphi_n = \varphi$, we immediately get the following theorem.

Theorem 5. Let <U, Form(U)> be the granular space on U, $\Sigma \subseteq$ Form(U) and $\varphi \in$ Form(U). The soundness on granular space holds, i.e. if $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$. □

We know that rules ($\in$), ($+$), ($\neg -$), ($\rightarrow +$), ($\rightarrow -$), ($\wedge +$), ($\wedge -$), ($\vee +$) and ($\vee -$) represent deduction patterns in some disciplines. So, formal reasoning describes the deduction processes occurring in the areas, such as mathematics, physics, computer science, etc. The soundness on granular space indicates that granular reasoning keeps the deduction processes. Now, we ask this question: does granular reasoning extend the deduction processes of formal reasoning? The following will give the answer to it.

B. Completeness on Granular Space

Let <U, Form(U)> be the granular space on U, $\Sigma \subseteq$ Form(U) and $\varphi \in$ Form(U). If $\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$, then the natural deduction system is said to have the property of completeness. Since the completeness is linked with the granular space <U, Form(U)>, we also call it the completeness on granular space.

However, the completeness on granular space does not hold. It can be explained by an example:

Example. Consider a universal set U={1, 2, 3}. Let P={<1, 2>, <1, 3>} and Q={<1, 2>,<1, 3>, <2, 3>}. P and Q are binary-relations on U. Let $\varphi = P(x_1, x_2)$ and $\psi = Q(x_1, x_2)$. Then $\varphi, \psi \in$ Form(U). By definition 4, we have $|\varphi| = P$ and $|\psi| = Q$. Since $P \subseteq Q$, we get $|\varphi| \subseteq |\psi|$, so $\varphi \models \psi$. But $\varphi \vdash \psi$ is not true, because we cannot use the rules in the natural deduction system to prove $\varphi \vdash \psi$. The form of formula φ is different from the form of formula ψ, and formal reasoning completely depends on formula's form.

As we have seen above, the soundness holds, but the completeness does not hold on granular space. This shows that formal reasoning can be copied by granular reasoning, but granular reasoning cannot be fully imitated by formal reasoning. Thus granular reasoning has extended the deduction patterns represented by the rules in the natural deduction system. From logical viewpoint, researchers always hope that a formal system not only has the property of soundness, but also has the property of completeness. However, the conclusions about the soundness and completeness on granular space show that granular reasoning is an extension of the deduction processes produced by the natural deduction system. This is just the result we are expecting, because we always try to set up reasoning approaches which develop classical reasoning.

The truth of the soundness on granular space is significant. This means that classical reasoning can be taken as a foundation to support granular reasoning, as well as to support decision reasoning. For any deduction process produced by the rules in the natural deduction system, granular reasoning or decision reasoning will preserve the deduction process.

V. Decision System's Decomposition

Granular reasoning can be used to study properties about a decision system. Based on granular reasoning, a decision system can be separated into sub-decision systems. We now discuss the decomposition approach which is closely connected with granular reasoning.

Consider a decision system $\boldsymbol{S}=(D, A_1 \cup A_2, V, f)$, which is represented by Table I. where $D=\{z_1, z_2, z_3, z_4\}$; $A_1=\{c_1, c_2\}$ and $A_2=\{d_1, d_2\}$, c_1 and c_2 are condition attributes, d_1 and d_2 are decision attributes; $V=\{1, 2, 3\}$; $f: D\times(A_1 \cup A_2) \rightarrow V$ is the information function, its correspondences are explicit, such as $f(z_1, c_1)=1$, $f(z_1, c_2)=2$, $f(z_2, d_1)=1$, $f(z_3, d_1)=2$, $f(z_3, d_2)=3$, etc. Sometimes, $\boldsymbol{S}=(D, A_1 \cup A_2, V, f)$ is abbreviated to $\boldsymbol{S}=(D, A_1 \cup A_2)$. As the discussion in section 1, we are able to get a decision rule:

$$(c_1, v_1)\wedge(c_2, v_2) \rightarrow (d_1, u_1)\wedge(d_2, u_2),$$

where $v_1, v_2, u_1, u_2 \in V$. This rule can be taken as a formula on $U=D \cup V$ (see the discussion between definition 4 and definition 5).

TABLE I.
DECISION SYSTEM S

D	c_1	c_2	d_1	d_2
z_1	1	2	1	2
z_2	1	2	1	2
z_3	2	3	2	3
z_4	2	3	3	3

TABLE II.
SUB-DECISION SYSTEM S_1

D	c_1	c_2	d_1
z_1	1	2	1
z_2	1	2	1
z_3	2	3	2
z_4	2	3	3

TABLE III.
SUB-DECISION SYSTEM S_2

D	c_1	c_2	d_2
z_1	1	2	2
z_2	1	2	2
z_3	2	3	3
z_4	2	3	3

Now, consider the granules $|(c_1, v_1)\wedge(c_2, v_2)|$ and $|(d_1, u_1)\wedge(d_2, u_2)|$. If $|(c_1, v_1)\wedge(c_2, v_2)| \subseteq |(d_1, u_1)\wedge(d_2, u_2)|$, by definition 6 we have $(c_1, v_1)\wedge(c_2, v_2) \models (d_1, u_1)\wedge(d_2, u_2)$, or by definition 0 we say that the decision rule $(c_1, v_1)\wedge(c_2, v_2) \rightarrow (d_1, u_1)\wedge(d_2, u_2)$ is true.

For the decision system $\boldsymbol{S}=(D, A_1\cup A_2)$, where $A_1=\{c_1, c_2\}$ and $A_2=\{d_1, d_2\}$, consider its two sub-decision systems $\boldsymbol{S}_1=(D, A_1\cup\{d_1\})$ and $\boldsymbol{S}_2=(D, A_1\cup\{d_2\})$, represented by Table II and Table III respectively, where $\{d_1\}\cup\{d_2\}=A_2$. Corresponding to $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, we have the decision rules $(c_1, v_1)\wedge(c_2, v_2)\rightarrow(d_1, u_1)$ and $(c_1, v_1)\wedge(c_2, v_2)\rightarrow(d_2, u_2)$. These decision rules have close links with the decision rule $(c_1, v_1)\wedge(c_2, v_2)\rightarrow(d_1, u_1)\wedge(d_2, u_2)$ corresponding to $\boldsymbol{S}$. In fact, from corollary 1, we get the following conclusion.

Corollary 2 $(c_1, v_1)\wedge(c_2, v_2)\models(d_1, u_1)\wedge(d_2, u_2)$ if and only if $(c_1, v_1)\wedge(c_2, v_2)\models(d_1, u_1)$ and $(c_1, v_1)\wedge(c_2, v_2)\models(d_2, u_2)$. □

Therefore, decision rule $(c_1, v_1)\wedge(c_2, v_2)\rightarrow(d_1, u_1)\wedge(d_2, u_2)$ is true if and only if decision rules $(c_1, v_1)\wedge(c_2, v_2)\rightarrow(d_1, u_1)$ and $(c_1, v_1)\wedge(c_2, v_2)\rightarrow(d_2, u_2)$ are true, simultaneously.

Hence, the decision reasoning based on decision systems $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ can determine the decision reasoning based on decision system $\boldsymbol{S}$; and vice versa. Using decision reasoning, we can separate decision system $\boldsymbol{S}=(D, A_1\cup\{d_1, d_2\})$ into sub-decision systems $\boldsymbol{S}_1=(D, A_1\cup\{d_1\})$ and $\boldsymbol{S}_2=(D, A_1\cup\{d_2\})$; also, we can combine $\boldsymbol{S}_1=(D, A_1\cup\{d_1\})$ and $\boldsymbol{S}_2=(D, A_1\cup\{d_2\})$ together to form $\boldsymbol{S}=(D, A_1\cup A_2)$. Granular reasoning sets up a bridge between a decision system and its sub-decision systems. This means that Table I can be divided into Table II and Table III; also, Table II and Table III can come together to form Table I, which are supported by granular reasoning.

More specifically, consider a decision rule of Table I:

$$(c_1, 1)\wedge(c_2, 2)\rightarrow(d_1, 1)\wedge(d_2, 2),$$

which is connected with decision rules $(c_1, 1)\wedge(c_2, 2)\rightarrow(d_1, 1)$ and $(c_1, 1)\wedge(c_2, 2)\rightarrow(d_2, 2)$ corresponding to Table II and Table III, respectively. It is not difficult to know that $(c_1, 1)\wedge(c_2, 2)\models(d_1, 1)\wedge(d_2, 2)$ if and only if $(c_1, 1)\wedge(c_2, 2)\models(d_1, 1)$ and $(c_1, 1)\wedge(c_2, 2)\models(d_2, 2)$.

Moreover, consider another decision rule of Table I:

$$(c_1, 2)\wedge(c_2, 3)\rightarrow(d_1, 3)\wedge(d_2, 3).$$

Corresponding to it, we get decision rule $(c_1, 2)\wedge(c_2, 3)\rightarrow(d_1, 3)$ of Table II, and decision rule $(c_1, 2)\wedge(c_2, 3)\rightarrow(d_2, 3)$ of Table III. From Table II, we know $|(c_1, 2)\wedge(c_2, 3)|=\{z_3, z_4\}$ and $|(d_1, 3)|=\{z_4\}$. Obviously, granule $|(c_1, 2)\wedge(c_2, 3)|$ is not contained in granule $|(d_1, 3)|$. This means that $(c_1, 2)\wedge(c_2, 3)\models(d_1, 3)$ fails to hold. It follows from corollary 2 that $(c_1, 2)\wedge(c_2, 3)\models(d_1, 3)\wedge(d_2, 3)$ does not hold. In fact, Table I shows $|(c_1, 2)\wedge(c_2, 3)|=\{z_3, z_4\}$ and $|(d_1, 3)\wedge(d_2, 3)|=\{z_4\}$. Since granule $|(c_1, 2)\wedge(c_2, 3)|$ is not contained in granule $|(d_1, 3)\wedge(d_2, 3)|$, $(c_1, 2)\wedge(c_2, 3)\models(d_1, 3)\wedge(d_2, 3)$ is not true.

Furthermore, let $\boldsymbol{S}=(D, A_1\cup\{d_1, d_2,\cdots, d_m\})$ be a decision system, here A_1 is the condition attribute set, and $d_1, d_2,\cdots, d_m$ are decision attributes. Based on granular reasoning, $\boldsymbol{S}=(D, A_1\cup\{d_1, d_2,\cdots, d_m\})$ can be divided into m sub-decision systems $\boldsymbol{S}_1=(D, A_1\cup\{d_1\})$, $\boldsymbol{S}_2=(D, A_1\cup\{d_2\})$,$\cdots$, and $\boldsymbol{S}_m=(D, A_1\cup\{d_m\})$. At the same time, $\boldsymbol{S}_1, \boldsymbol{S}_2,\cdots$, and $\boldsymbol{S}_m$ can come together to form $\boldsymbol{S}$ as well.

Thus, when making decision reasoning, it is sufficient to consider such decision systems which have only one decision attribute.

Granular reasoning provides theoretical support for the decomposition and combination of decision systems. This is an application of granular reasoning.

VI. Conclusion

Granular reasoning introduced in this paper is a generalized form of decision reasoning. The researches on it focus on two aspects. First, we investigate the connections between granular reasoning and classical reasoning. The related theorems show that granular reasoning satisfies the granular patterns of reasoning which correspond to the rules in the natural deduction system. This is significant because we conclude from these theorems that granular reasoning can imitate the deduction processes occurring in classical logic, at the same time, granular reasoning has extended the deduction processes. Second, we use granular reasoning to divide a decision system into sub-decision systems. Therefore, decision reasoning which is based on a decision system becomes to rely on sub-decision systems. This makes decision reasoning easier. All of these constitute the main part of our researches in this paper.

The idea of introducing and studying granular reasoning stems from decision logic[1]. We are aimed at laying a foundation for decision reasoning. The analysis shows that decision reasoning is a special form of granular reasoning, and granular reasoning covers a wider range than decision reasoning. This indicates that the granular patterns of reasoning are also satisfied by decision reasoning. Hence, there are close connections between classical reasoning and decision reasoning. Classical logic provides support for decision logic.

One of the purposes of making researches on granular reasoning is to study granular computing[7-12] that is a current research topic. Informally, granular computing can be regarded as various operations, combinations or relations which correlate with granules[2]. From definition 6 we know that granular reasoning is defined by the inclusion relation over granules, and operations or combinations of granules are used to decide whether a granule is contained in another. Thus, granular reasoning can be thought of as one kind of granular computing. This can be clearly seen from the proofs of the theorems in section 3.3, in which intersections or unions of granules are frequently used to get some consequents. So, granular reasoning provides an approach for the study of granular computing.

Also, the applications of granular reasoning is an important aspect of our study. As an example of applied researches, an application is discussed in section 5, which shows that granular reasoning can be used to separate or combine decision systems. However, whether granular reasoning can be applied in a wide range deserves further consideration in the future.

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REFERENCES

[1] Z. Pawlak, *Rough Set—Theoretical Aspects of Reasoning about Data*, Dordrecht, Holland: Kluwer Academic Publishers, 1992.
[2] Yan Lin, Liu Qing, "A logical method of formalization for granular computing", The Proceedings of 2007 IEEE International Conference on Granular Computing, Silicon Valley, California, USA, 2007, pp.22-27.
[3] A. Nakamura, M. J. Gao, "A rough logic based on incomplete information and its application", International Journal of Approximate Reasoning, vol.15, No.4, 1996, pp.367-378.
[4] Z. Pawlak, "Rough logic", Bulletin of Polish Academy of Sciences Technical Sciences, vol.35, No.5-6, 1987, pp.253-258.
[5] Y. Y. Yao, Q. Liu, "A generalized decision logic in interval-set-valued information table", Lecture Notes in Artificial Intelligence 1711, Berlin, Germany: Springer-Verlag, 1999, pp.285-294.
[6] B. Kolman, R. C. Busby, S. C. Ross, *Discrete Mathematical Structures*(fourth edition), New Jersey, USA: Prentice-Hall, 2001.
[7] M. J. Wierman, "Measuring uncertainty in rough set theory", International Journal of General Systems, vol.28, No.4, 1999, pp.283-297.
[8] Yan Lin, *Fundamentals of Mathematical Logic and Granular Computing*, Beijing, China: Science Press, 2007(in Chinese).
[9] T. Y. Lin, "Granular computing", Lecture Notes in Computer Science 2639, Berlin, Germany: Springer-Verlag, 2003, pp.16-24.
[10] T. Y. Lin, "Granular computing on binary relations Ⅱ: rough set representations and belief functions", L. Polkowsk, A. Skowron eds, Rough Sets in Knowledge Discovery, Heidelberg, Germany: Physica-Verlag, 1998, pp.121-140.
[11] Y. Y. Yao, Y. Zhao, "Attribute reduction in decision-theoretic rough set methods", Information Sciences, vol.178, 2008, pp.3356-3373.
[12] Z. Pawlak, "Rough sets and intelligent data analysis", Information Sciences, vol.147, No.1-4, 2002, pp.1-12.

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