Intuitionistic fuzzy dominance–based rough set approach: model and attribute reductions

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Abstract—The dominance-based rough set approach plays an important role in the development of the rough set theory. It can be used to express the inconsistencies coming from consideration of the preference-ordered domains of the attributes. The purpose of this paper is to further generalize the dominance-based rough set model to fuzzy environment. The constructive approach is used to define the intuitionistic fuzzy dominance-based lower and upper approximations respectively. Basic properties of the intuitionistic fuzzy dominance-based rough approximations are then examined. By introducing the concept of approximate distribution reducts into intuitionistic fuzzy dominancebased rough approximations, four different forms of reducts are defined. The judgment theorems and discernibility matrixes associated with these reducts are also obtained. Such results are all intuitionistic fuzzy generalizations of the classical dominance-based rough set approach. Some numerical examples are employed to substantiate the conceptual arguments.

Index Terms—dominance–based rough set, dominance– based fuzzy rough set, intuitionistic fuzzy dominance relation, intuitionistic fuzzy dominance–based rough set, approximate distribution reducts

I. INTRODUCTION

Rough set theory [44]–[47], proposed by Pawlak, is a new mathematical tool which can be used to deal with vague and uncertain information. The lower and upper approximate operators are key notions in rough set theory, they were constructed on the basis of an indiscernibility relation (equivalence relation, i.e. reflexive, symmetric and transitive). By using such two approximations, knowledge hidden in the information tables may be unravelled and expressed in the form of decision rules.

It is well known that the indiscernibility relation is too restrictive for classification analysis in practical applications. Therefore, many authors have generalized the notions of rough approximations by using some more general binary relations, e.g., tolerance relation [35], [37], similarity relation [52]-[54], characteristic relation [27], [28], [39], etc. These extensions of the rough approximations may be used on reasoning and acquisition of knowledge in incomplete systems [38], [48], [58], [61], [63], [66]-[68], continues-valued systems [29], [56] and some more complex forms of information systems. Moreover, it should be noticed that the generalization of rough approximations to fuzzy environments also plays an important role in the development of rough set theory. For example, in Ref. [17], the model of rough fuzzy set was proposed by using the indiscernibility relation to approximate a fuzzy concept. Alternatively, the fuzzy rough model is the approximation of a crisp set or a fuzzy set in a fuzzy approximation space. In Ref. [55], Sun et al. presented the interval-valued fuzzy rough set by combining the interval-valued fuzzy set and rough set. By employing an approximation space constituted from an intuitionistic fuzzy triangular norm, an intuitionistic fuzzy implicator, and an intuitionistic fuzzy T-equivalence relation, Cornelis et al. [12] defined the concept of intuitionistic fuzzy rough sets in which the lower and upper approximations are both intuitionistic fuzzy sets on the universe of discourse. Zhou et al. proposed the intuitionistic fuzzy rough approximation from the viewpoint of constructive and axiomatic approaches respectively in Ref. [75]. Bhatt [4] presented the fuzzy-rough sets on compact computational domain. Zhao et al. [74] investigated the fuzzy variable precision rough sets by combining fuzzy rough set and variable precision rough set with the goal of making fuzzy rough set a special case. Hu et al. [31] proposed the Gaussian Kernel based fuzzy rough set, it uses the Gaussian Kernel to compute fuzzy T-equivalence relation for objective approximation. The same authors also proposed the fuzzy preference-based rough sets in Ref. [32]. Ouyang et al. [43] presented a fuzzy rough model, which is based on the fuzzy tolerance relation. More details about recent advancements of fuzzy rough set can be found in the literatures [9], [10], [16], [40], [41], [57], [62], [69], [70], [72].

On the other hand, though the rough set has been

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demonstrated to be useful in the fields of knowledge discovery, decision analysis, pattern recognition and so on, it is not able, however, to discover inconsistencies coming from consideration of criteria, that is, attributes with preference-ordered domains, such as product quality, market share and debt ratio. To solve this problem, Greco et al. have proposed an extension of Pawlak's rough set approach, which is called the Dominance-based Rough Set Approach (DRSA) [5]-[8], [13]-[15], [18]-[26], [30], [34], [71]. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. Presently, work on dominance-based rough set model also progressing rapidly. For example, by considering two different types of semantic explanations of unknown values, Shao et al. and Yang et al. generalized the DRSA to incomplete environments in Ref. [50] and Ref. [65] respectively. Wei et al. presented the concept of valued dominance-based rough approximations in Ref. [60]. With introduction of the concept of variable precision rough set [76] into DRSA, Błszczyńki et al. proposed the variable consistency dominance-based rough set approach [6], [8], Inuiguchi et al. proposed the variable precision dominance-based rough set [33]. Kotłowski et al. [34] introduced a new approximation of DRSA which is based on the probabilistic model for the ordinal classification problems. Greco et al. generalized the DRSA to fuzzy environment and then presented the model of dominance-based rough fuzzy set in Ref. [20]. By using a fuzzy dominance relation, the same authors also presented dominance-based fuzzy rough set [26] in their literatures.

As a generalization of the Zadeh fuzzy set, the notion of intuitionistic fuzzy set was suggested for the first time by Atanassov [1], [2]. An intuitionistic fuzzy set allocates to each element both a degree of membership and one of non-membership, and it was applied to the fields of approximate inference, signal transmission and controller, etc. In this paper, the intuitionistic fuzzy set will be combined with the DRSA and then the model of Intuitionistic Fuzzy Dominance-based Rough Set(IFDRS) is presented. The IFDRS is a new generalization of the classical DRSA because we use an intuitionistic fuzzy dominance relation instead of the crisp or fuzzy dominance relation to approximate the upward and downward unions of the decision classes. It should be noticed here that we use the constructive approach to define the IFDRS in this paper.

In traditional DRSA, the dominance relation can only be used to judge whether an object is dominating another one. Furthermore, to express the credibility that an object is dominating another one, the fuzzy dominance relation is then presented (eg. Ref. [26] and Ref. [60]). In fuzzy dominance relation, if an object x dominates another object y with a credibility α , then it naturally follows that x does not dominate y to the extent $1 - \alpha$. To further generalize such idea, it is naturally to introduce the intuitionistic fuzzy approach into DRSA, i.e. IFDRS. In our IFDRS, the intuitionistic fuzzy dominance relation can express not only the credibility that x dominates y, but also the non-credibility that x dominates y.

Once a new rough set model is presented, the immediate problem is attribute reduction. It involves the search for particular subsets of attributes, which provide the same information for some purpose as the full set of available attributes. Such subsets are called reducts. In traditional rough set theory, Pawlak proposed the positive-region based reduct, which can be used to preserve the union of all lower approximations. Following Pawlak's work, Kryszkiewicz [36] investigated and compared five notions of knowledge reductions in inconsistent systems, Zhang et al. [73] proposed the concepts of distribution reduct and maximal distribution reduct. Moreover, Wang et al. [59] presented a systematic approach to knowledge reduction which is based on the general binary relation and the corresponding rough approximation, Chen et al. [11] investigated the problem of knowledge reduction in decision system with covering based rough approximation, Yang et al. [64] constructed a new reduction theory by redefining the approximation space and the reducts of covering generalized rough set. By using the variable precision rough set model [76], Beynon [3] proposed the concept of β -reduct, Mi et al. proposed the lower and upper approximate distribution reducts in Ref. [42].

In this paper, we will further introduce Mi's approximate distribution reducts into our IFDRS. Four notions of approximate distribution reducts are then presented because there are two pairs of approximations in DRSA. The judgment theorems and discernibility matrices associated with these reducts are also established, from which we obtain the practical approaches to compute approximate distribution reducts in IFDRS.

To facilitate our discussion, we first present basic notions of classical DRSA and dominance–based fuzzy rough set in Section 2. The constructive approach to define IFDRS is presented in Section 3. We also employ an illustrative example to show how the IFDRS can be used in decision system with probabilistic interpretation. In Section 4, the approximate distribution reducts in terms of our IFDRS are investigated. Results are summarized in Section 5.

II. DOMINANCE-BASED ROUGH SET APPROACH

A. Greco's DRSA

A decision system is a pair $\mathscr{I} = < U, AT \cup \{d\} >,$ where

- U is a non-empty finite set of objects, it is called the universe;
- *AT* is a non–empty finite set of conditional attributes;
- d is the decision attribute where $AT \cap \{d\} = \emptyset$.

 $\forall a \in AT, V_a \text{ is used to represent the domain of attribute } a \text{ and then } V = V_{AT} = \bigcup_{a \in AT} V_a \text{ is the domain of all attributes. Moreover, for each } x \in U$, let us denote by a(x) the value that x holds on $a \ (a \in AT)$.

By considering the preference-ordered domains of attributes (criteria), Greco et al. have proposed an extension of the classical rough set that is able to deal with inconsistencies typical to exemplary decisions in Multi-Criteria Decision Making (MCDM) problems, which is called the Dominance-based Rough Set Approach (DRSA). let \succeq_a be a weak preference relation on U (often called outranking) representing a preference on the set of objects with respect to criterion $a \ (a \in AT)$; $x \succeq_a y$ means "x is at least as good as y with respect to criterion a". We say that x dominates y with respect to $A \subseteq AT$, iff $x \succeq_a y$ for each $a \in A$.

By the above discussion, we can define the following two sets for each object x in \mathscr{I} such that:

- the set of objects dominate x, i.e. $[x]_{4}^{\geq} = \{y \in U :$ $\forall a \in A, y \succeq_a x \};$
- the set of objects dominated by x, i.e. $[x]_A^{\leq} = \{y \in$ $U: \forall a \in A, x \succeq_a y \}.$

In the traditional DRSA, we assume here that the decision attribute d determines a partition of U into a finite number of classes; let $\mathbf{CL} = \{CL_n, n \in N\}, N =$ $\{1, 2, \cdots, m\}$, be a set of these classes that are ordered. Different from Pawlak's rough approximation, in DRSA, the sets to be approximated are an upward union and a downward union of decision classes, which are defined respectively as $CL_n^{\geq} = \bigcup_{n' \geq n} CL_{n'}, CL_n^{\leq} = \bigcup_{n' \leq n} CL_{n'},$

 $n, n' \in N.$

In Greco's DRSA, the A-lower approximation and Aupper approximation of CL_n^{\geq} are:

$$\underline{A}(CL_n^{\geq}) = \{x \in U : [x]_A^{\geq} \subseteq CL_n^{\geq}\},\$$
$$\overline{A}(CL_n^{\geq}) = \{x \in U : [x]_A^{\leq} \cap CL_n^{\geq} \neq \emptyset\}.$$

the A-lower approximation and A-upper approximation of CL_n^{\leq} are:

$$\underline{A}(CL_n^{\leq}) = \{ x \in U : [x]_A^{\leq} \subseteq CL_n^{\leq} \},$$

$$\overline{A}(CL_n^{\leq}) = \{ x \in U : [x]_A^{\geq} \cap CL_n^{\leq} \neq \emptyset \};$$

the A-boundaries of CL_n^{\geq} and CL_n^{\leq} are:

$$BN_A(CL_n^{\geq}) = \overline{A}(CL_n^{\geq}) - \underline{A}(CL_n^{\geq}),$$

$$BN_A(CL_n^{\leq}) = \overline{A}(CL_n^{\leq}) - \underline{A}(CL_n^{\leq}).$$

B. Dominance-based fuzzy rough set

Dominance-based fuzzy rough set is a fuzzy generalization of DRSA. In dominance-based fuzzy rough set model, the dominance relation is replaced by a fuzzy dominance relation.

Definition 1: Let R_a be a fuzzy dominance relation on U with respect to attribute a, i.e. $R_a: U \times U \rightarrow [0,1]$, $\forall x, y \in U, R_a(x, y)$ represents the credibility of the proposition "x is at least as good as y with respect to attribute a". A fuzzy dominance relation on U (denotation $R_A(x,y)$) can be defined for each $A \subseteq AT$ as:

$R_A(x,y) = \wedge \{R_a(x,y) : a \in A\}.$

Definition 2: Let \mathscr{I} be a decision system in which $A \subseteq AT, \forall n \in N$, the A-lower approximation and A-upper approximation of CL_n^{\geq} with respect to fuzzy dominance relation are denoted by $A_R(CL_n^{\geq})$ and $\overline{A_R}(CL_n^{\geq})$ respectively, whose memberships for each $x \in$ U, are defined as:

$$\mu_{\underline{A_R}(CL_n^{\geq})}(x) = \wedge_{y \in U} \left(\mu_{CL_n^{\geq}}(y) \lor (1 - R_A(y, x)) \right)$$

$$\mu_{\overline{A_R}(CL_n^{\geq})}(x) = \vee_{y \in U} \left(\mu_{CL_n^{\geq}}(y) \land R_A(x, y) \right)$$

 $\forall n \in N$, the A-lower approximation and A-upper approximation of $CL_{\overline{n}}^{\leq}$ with respect to fuzzy dominance relation are denoted by $A_R(CL_{\overline{n}}^{\leq})$ and $\overline{A_R}(CL_{\overline{n}}^{\leq})$ respectively, whose memberships for each $x \in U$, are defined as:

$$\mu_{\underline{A}_{R}(CL_{n}^{\leq})}(x) = \wedge_{y \in U} \left(\mu_{CL_{n}^{\leq}}(y) \vee (1 - R_{A}(x, y)) \right)$$

$$\mu_{\overline{A_R}(CL_n^{\leq})}(x) = \bigvee_{y \in U} (\mu_{CL_n^{\leq}}(y) \land R_A(y, x))$$

More details about the dominance-based fuzzy rough set can be found in Ref. [26].

III. INTUITIONISTIC FUZZY DOMINANCE-BASED ROUGH SET

A. Construction of intuitionistic fuzzy dominance-based rough sets

An intuitionistic fuzzy set \mathscr{F} in U is given by

 $\mathscr{F} = \{ \langle x, u_{\mathscr{F}}(x), v_{\mathscr{F}}(x) \rangle : x \in U \}$

where $u_{\mathscr{F}}: U \to [0,1]$ and $v_{\mathscr{F}}: U \to [0,1]$ with the condition such that $0 \leq u_{\mathscr{F}}(x) + v_{\mathscr{F}}(x) \leq 1$. The numbers $u_{\mathscr{F}}(x), v_{\mathscr{F}}(x) \in [0,1]$ denote the degree of membership and non-membership of x to \mathscr{F} , respectively. Obviously, when $u_{\mathscr{F}}(x) + v_{\mathscr{F}}(x) = 1$, for all elements in the universe, the traditional fuzzy set concept is recovered. The family of all intuitionistic fuzzy subsets on U is denoted by $\mathscr{IF}(U)$. Let us review some basic operations on $\mathscr{IF}(U)$ as follows:

$$\forall \mathscr{F}_1, \mathscr{F}_2 \in \mathscr{IF}(l)$$

1)
$$U - \mathscr{F}_1 = \{ \langle x, v_{\mathscr{F}_1}(x), u_{\mathscr{F}_1}(x) \rangle : x \in U \};$$

2)
$$\mathscr{F}_1 \wedge \mathscr{F}_2 = \{ \langle x, u_{\mathscr{F}_1}(x) \wedge u_{\mathscr{F}_2}(x), v_{\mathscr{F}_1}(x) \lor v_{\mathscr{F}_2}(x) \rangle : x \in U \};$$

- 3) $\mathscr{F}_1 \vee \mathscr{F}_2 = \{\langle x, u_{\mathscr{F}_1}(x) \vee u_{\mathscr{F}_2}(x), v_{\mathscr{F}_1}(x) \land v_{\mathscr{F}_2}(x) \rangle : x \in U\};$ 4) $\mathscr{F}_1 \subseteq \mathscr{F}_2 \Leftrightarrow u_{\mathscr{F}_1}(x) \leq u_{\mathscr{F}_2}(x), v_{\mathscr{F}_1}(x) \geq$
- $v_{\mathscr{F}_2}(x), \forall x \in U;$

5)
$$\mathscr{F}_1 \supseteq \mathscr{F}_2 \Leftrightarrow \mathscr{F}_2 \subseteq \mathscr{F}_1;$$

6) $\mathscr{F}_1 = \mathscr{F}_2 \Leftrightarrow \mathscr{F}_1 \subseteq \mathscr{F}_2, \mathscr{F}_1 \supseteq \mathscr{F}_2.$

By the definition of intuitionistic fuzzy set, we know that an intuitionistic fuzzy relation \mathscr{R} on U is an intuitionistic fuzzy subset of $U \times U$, namely, \mathscr{R} is given by

$$\mathscr{R} = \{ < (x, y), u_{\mathscr{R}}(x, y), v_{\mathscr{R}}(x, y) >: (x, y) \in U \times U > \}$$

where

$$u_{\mathscr{R}}: U \times U \to [0,1]$$
 and $v_{\mathscr{R}}: U \times U \to [0,1]$

satisfy with the condition $0 \le u_{\mathscr{R}}(x,y) + v_{\mathscr{R}}(x,y) \le 1$ for each $(x, y) \in U \times U$. The set of all intuitionistic fuzzy relation on U is denoted by $\mathscr{IFR}(U \times U)$.

Definition 3: Let U be the universe of discourse, $\forall \mathscr{R} \in$ $\mathscr{IFR}(U \times U),$ if

- 1) $u_{\mathscr{R}}(x,y)$ represents the credibility of the proposition "x is at least as good as y in \mathscr{R} ";
- 2) $v_{\mathscr{R}}(x,y)$ represents the non-credibility of the proposition "x is at least as good as y in \mathscr{R} ";

then \mathscr{R} is referred to as an intuitionistic fuzzy dominance relation

By the above definition, we can see that different from the fuzzy dominance relation we used in Section 2.2, intuitionistic fuzzy dominance relation can express not only the credibility of dominance principle between different objects, but also the non-credibility of dominance principle between these objects. In a decision system, suppose that for each $a \in AT$, we have an intuitionistic fuzzy dominance relation \mathcal{R}_a , then the intuitionistic fuzzy dominance relation in terms of AT is denoted by \mathscr{R}_{AT} where

$$\mathcal{R}_{AT}(x,y) = \langle u_{\mathcal{R}_A}(x,y), v_{\mathcal{R}_A}(x,y) \rangle$$

= $\langle \wedge \{ u_{\mathcal{R}_a}(x,y) : a \in AT \},$
 $\vee \{ v_{\mathcal{R}_a}(x,y) : a \in AT \} \rangle$ (1)

for each $(x, y) \in U \times U$. To simplify our discussion, the intuitionistic fuzzy dominance relation we used in this paper is always reflexive, i.e. $\mathscr{R}_a(x,x) = 1$ $(u_{\mathscr{R}_a}(x,x) =$ 1, $v_{\mathscr{R}_a}(x, x) = 0$) for each $x \in U$ and each $a \in AT$.

Definition 4: Let I be a decision system in which $A \subseteq AT, \mathscr{R}_A$ is an intuitionistic fuzzy dominance relation with respect to A, $\forall n \in N$, the A-lower approximation and A-upper approximation of CL_n^{\geq} with respect to intuitionistic fuzzy dominance relation \mathscr{R}_A are denoted by $A_{\mathscr{R}}(CL_{\overline{n}}^{\geq})$ and $\overline{A_{\mathscr{R}}}(CL_{\overline{n}}^{\geq})$, respectively and

 $\overline{A_{\mathscr{R}}}(CL_{n}^{\geq}) = \{ \langle x, u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x), v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x) \rangle : x \in U \}$

 $u_{A_{\mathscr{R}}(CL^{\geq}_{\pi})}(x) = \wedge_{y \in U} \left(u_{CL^{\geq}_{\pi}}(y) \vee v_{\mathscr{R}_{A}}(y,x) \right);$

 $v_{\mathcal{A}_{\mathscr{R}}(CL_{\pi}^{\geq})}(x) = \forall_{y \in U} \left(v_{CL_{\pi}^{\geq}}(y) \wedge u_{\mathscr{R}_{A}}(y, x) \right);$

 $u_{\overline{A\mathscr{R}}(CL^{\geq})}(x) = \forall_{y \in U} \left(u_{CL^{\geq}}(y) \wedge u_{\mathscr{R}_A}(x,y) \right);$

 $v_{\overline{A\mathscr{R}}(CL^{\geq}_{\pi})}(x) = \wedge_{y \in U} (v_{CL^{\geq}_{\pi}}(y) \vee v_{\mathscr{R}_{A}}(x,y));$

the A-lower approximation and A-upper approximation

of $CL_{\overline{n}}^{\leq}$ with respect to intuitionistic fuzzy dominance relation \mathscr{R}_A are denoted by $\underline{A}_{\mathscr{R}}(CL_n^{\leq})$ and $\overline{A}_{\mathscr{R}}(CL_n^{\leq})$,

the 1) $u_{A_{\mathscr{R}}(CL_{n}^{\geq})}(x)(v_{A_{\mathscr{R}}(CL_{n}^{\geq})}(x))$ is membership(non-membership) of xbelongs to the lower approximation $A_{\mathscr{R}}(CL_n^{\geq})$;

- 2) $u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x)(v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x))$ is membership(non-membership) of x the belongs to the upper approximation $\overline{A_{\mathscr{R}}}(CL_{\overline{n}}^{\geq})$;
- 3) $u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x)(v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x))$ membership(non-membership) the is of x belongs to the lower approximation $A_{\mathscr{R}}(CL_{\overline{n}}^{\leq})$;
- $\frac{u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\leq})}(x)(v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\leq})}(x))}{\text{membership(non-membership)}}$ the 4) is of x belongs to the upper approximation $\overline{A_{\mathscr{R}}}(CL_{\overline{n}}^{\leq})$.

Since mathematically, an intuitionistic fuzzy set may be equivalently characterized by an interval-valued fuzzy set, then our intuitionistic fuzzy dominance-based rough set models can also be considered as a type of interval-valued fuzzy rough set model. However, it should be noticed that our IFDRS is different from the common interval-valued fuzzy rough set because the intuitionistic fuzzy dominance relation we used here has its own semantic explanation, i.e. it represents both the credibility and non-credibility of the dominance principle.

Theorem 1: Let \mathscr{I} be a decision system in which $A \subseteq$ AT, \mathscr{R}_A is the intuitionistic fuzzy dominance relation related to A, if $u_{\mathscr{R}_A}(x,y) + v_{\mathscr{R}_A}(x,y) = 1$ for each $(x,y) \in U \times U$, then for each $x \in U$, we have

- 1) $u_{A_{\mathscr{R}}(CL_{n}^{\geq})}(x) + v_{A_{\mathscr{R}}(CL_{n}^{\geq})}(x) = 1;$
- 2) $u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}^{\underline{a}}(x) + v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}^{\underline{a}}(x) = 1;$
- 3) $u_{\underline{A}_{\mathscr{R}}(CL_{n}^{d})}(x) + v_{\underline{A}_{\mathscr{R}}(CL_{n}^{d})}(x) = 1;$
- 4) $u_{\overline{A_{\mathscr{R}}}(CL_{\overline{n}}^{\leq})}(x) + v_{\overline{A_{\mathscr{R}}}(CL_{\overline{n}}^{\leq})}(x) = 1.$

Proof: We only prove (1), the proofs of (2), (3) and $\underline{A}_{\mathscr{R}}(CL_{n}^{\geq}) = \{ \langle x, u_{A_{\mathscr{R}}(CL_{n}^{\geq})}(x), v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) \rangle : x \in U \}, (4) \text{ are similar to the proof of (1)}.$

$$\forall x \in U, \text{ by Definition 4, we have } u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) + v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) = \wedge_{y \in U} (u_{CL_{n}^{\geq}}(y) \lor v_{\mathscr{R}_{A}}(y,x)) + \bigvee_{y \in U} (v_{CL_{n}^{\geq}}(y) \land u_{\mathscr{R}_{A}}(y,x)).$$

$$\forall y \in U,$$

• If $y \in CL_n^{\geq}$, i.e. $u_{CL_n^{\geq}}(y) = 1$ and $v_{CL_n^{\geq}}(y) = 0$, then $(u) \lor v_{\mathscr{R}_A}(y, x) = 1,$

$$\begin{split} & u_{CL_n^{\geq}}(y) \lor v_{\mathscr{R}_A}(y,x) \equiv 1, \\ & v_{CL_n^{\geq}}(y) \land u_{\mathscr{R}_A}(y,x) = 0. \end{split}$$

• If $y \notin CL_n^{\geq}$, i.e. $u_{CL_n^{\geq}}(y) = 0$ and $v_{CL_n^{\geq}}(y) = 1$, then

 $\notin CL_n^{\geq}$ such

$$u_{CL_n^{\geq}}(y) \lor v_{\mathscr{R}_A}(y, x) = v_{\mathscr{R}_A}(y, x),$$

$$v_{CL_n^{\geq}}(y) \wedge u_{\mathscr{R}_A}(y, x) = u_{\mathscr{R}_A}(y, x).$$

$$\begin{split} u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x) &= \wedge_{y \in U} (u_{CL_{n}^{\leq}}(y) \lor v_{\mathscr{R}_{A}}(x,y)); \\ v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x) &= \lor_{y \in U} (v_{CL_{n}^{\leq}}(y) \land u_{\mathscr{R}_{A}}(x,y)); \\ u_{\overline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x) &= \lor_{y \in U} (u_{CL_{n}^{\leq}}(y) \land u_{\mathscr{R}_{A}}(y,x)); \\ v_{\overline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x) &= \wedge_{y \in U} (v_{CL_{n}^{\leq}}(y) \lor v_{\mathscr{R}_{A}}(y,x)). \\ By \text{ the above definition, we know that} \end{split}$$

 $U\times U, \, {\rm then}\,\, u_{\underline{A}_{\mathscr{R}}(CL_n^{\geq})}(x) + v_{\underline{A}_{\mathscr{R}}(CL_n^{\geq})}(x) = v_{\mathscr{R}_A}(y,x) + \dots + v_{\mathcal{R}}(y,x) + \dots + v_{\mathcal{R}}(y,x) = v_{\mathcal{R}}(y,x) + \dots + v_{\mathcal{R}}(y,x) + \dots + v_{\mathcal{R}}(y,x) = v_{\mathcal{R}}(y,x) + \dots + v_{\mathcal{R}}(y,x) + \dots + v_{\mathcal{R}}(y,x) = v_{\mathcal{R}}(y,x) + \dots + v_{\mathcal{R}}(y,x) = v_{\mathcal{R}}$ $u_{\mathscr{R}_A}(y,x)$ = 1

Since $u_{\mathscr{R}_A}(x,y) + v_{\mathscr{R}_A}(x,y) = 1$ for each $(x,y) \in$

By the above theorem, we can see that if the intuitionistic fuzzy dominance relation degenerated to be the

where

respectively and

fuzzy dominance relation, then the intuitionistic fuzzy dominance-based rough set we showed in Definition 4 will degenerate to be the fuzzy dominance-based rough set. From this point of view, the intuitionistic fuzzy dominance-based rough set is a generalization of the traditional fuzzy dominance-based rough set.

Theorem 2: Let \mathscr{I} be a decision system in which $A \subseteq$ AT, we have

- 1) $u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) = \wedge \{v_{\mathscr{R}_{A}}(y,x) : y \notin CL_{n}^{\geq}\} (n =$ $2, \overline{\cdots}, m$;
- 2) $v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) = \lor \{u_{\mathscr{R}_{A}}(y,x) : y \notin CL_{n}^{\geq}\} (n = 2, \cdots, m);$
- 3) $u_{\overline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) = \vee \{u_{\mathscr{R}_{A}}(x,y) : y \in CL_{n}^{\geq}\} (n = 1, \cdots, m);$
- 4) $v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x) = \wedge \{v_{\mathscr{R}_{A}}(x,y) : y \in CL_{n}^{\geq}\} (n = 1, \cdots, m);$
- 5) $u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x) = \wedge \{v_{\mathscr{R}_{A}}(x,y) : y \notin CL_{n}^{\leq}\} \ (n =$ $1, \overline{\cdots}, m-1);$
- 6) $v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x) = \forall \{u_{\mathscr{R}_{A}}(x,y) : y \notin CL_{n}^{\leq}\} \ (n = 1)$
- $\begin{array}{l} 1, \cdots, m-1;\\ 1, \cdots, m-1;\\ 1 \end{array} \quad u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\leq})}(x) = \lor \{u_{\mathscr{R}_{A}}(y, x) : y \in CL_{n}^{\leq}\} \ (n=1, \cdots, m); \end{array}$
- 8) $v_{\overline{A\mathscr{R}}(CL_n^{\leq})}(x) = \wedge \{ v_{\mathscr{R}_A}(y, x) : y \in CL_n^{\leq} \} \ (n = 1, \cdots, m).$

Proof: We only prove 1), others can be proved analogously.

 $\forall n = 2, \cdots, m$, by Definition 4, we have

$$u_{\underline{A}_{\underline{\mathscr{R}}}(CL_{n}^{\geq})}(x) = \wedge_{y \in U} \big(u_{CL_{n}^{\geq}}(y) \vee v_{\mathscr{R}_{A}}(y,x) \big).$$

- If $y \in CL_n^{\geq}$, then $u_{CL_n^{\geq}}(y) = 1$, it follows that $u_{CL_n^{\geq}}(y) \vee v_{\mathscr{R}_A}(y,x) = 1;$
- If $y \notin CL_n^{\geq}$, then $u_{CL_n^{\geq}}(y) = 0$, it follows that $u_{CL_n^{\geq}}(y) \vee v_{\mathscr{R}_A}(y,x) \stackrel{\sim}{=} v_{\mathscr{R}_A}(y,x).$

Since $n = 2, \dots, m$, then there must be $y \notin CL_{\overline{n}}^{\geq}$ such that $u_{CL^{\geq}_{n}}(y) \vee v_{\mathscr{R}_{A}}(y,x) = v_{\mathscr{R}_{A}}(y,x)$. From discussion above, it is not difficult to conclude that $u_{A_{\mathscr{R}}(CL^{\geq}_n)}(x) =$ $\wedge \{ v_{\mathscr{R}_A}(y, x) : y \notin CL_n^{\geq} \}.$

Theorem 3: Let I be a decision system in which $A \subseteq AT$, the intuitionistic fuzzy dominance–based rough approximations have the following properties:

1) Contraction and extension:

$$\frac{A_{\mathscr{R}}(CL_{n}^{\geq}) \subseteq CL_{n}^{\geq} \subseteq \overline{A_{\mathscr{R}}(CL_{n}^{\geq})};}{A_{\mathscr{R}}(CL_{n}^{\leq}) \subseteq CL_{n}^{\leq} \subseteq \overline{A_{\mathscr{R}}(CL_{n}^{\leq})};}$$

2) Complements:

$$\begin{split} \underline{A}_{\mathscr{R}}(CL_{n}^{\geq}) &= U - \overline{A}_{\mathscr{R}}(CL_{n-1}^{\leq}), n = 2 \cdots m \\ \underline{A}_{\mathscr{R}}(CL_{n}^{\leq}) &= U - \overline{A}_{\mathscr{R}}(CL_{n+1}^{\geq}), n = 1 \cdots m - 1 \\ \overline{A}_{\mathscr{R}}(CL_{n}^{\geq}) &= U - \underline{A}_{\mathscr{R}}(CL_{n-1}^{\leq}), n = 2 \cdots m \\ \overline{A}_{\mathscr{R}}(CL_{n}^{\leq}) &= U - \underline{A}_{\mathscr{R}}(CL_{n+1}^{\leq}), n = 1 \cdots m - 1 \end{split}$$

3) Monotones with attributes:

$$\frac{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq}) \subseteq \underline{AT}_{\mathscr{R}}(CL_{n}^{\geq}); A_{\mathscr{R}}(CL_{n}^{\geq}) \supseteq AT_{\mathscr{R}}(CL_{n}^{\geq});}{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq}) \subseteq \underline{AT}_{\mathscr{R}}(CL_{n}^{\leq}); \overline{A}_{\mathscr{R}}(CL_{n}^{\leq}) \supseteq \overline{AT}_{\mathscr{R}}(CL_{n}^{\leq});}$$

4) Monotones with decision classes:

 $n_1, n_2 \in N$ such that $n_1 \leq n_2$

$$\underline{\underline{A}_{\mathscr{R}}}(CL_{n_{1}}^{\geq}) \supseteq \underline{\underline{A}_{\mathscr{R}}}(CL_{n_{2}}^{\geq}); \overline{\underline{A}_{\mathscr{R}}}(CL_{n_{1}}^{\geq}) \supseteq \overline{\underline{A}_{\mathscr{R}}}(CL_{n_{2}}^{\geq}); \\ \underline{\underline{A}_{\mathscr{R}}}(CL_{n_{1}}^{\leq}) \subseteq \underline{\underline{A}_{\mathscr{R}}}(CL_{n_{2}}^{\leq}); \overline{\underline{A}_{\mathscr{R}}}(CL_{n_{1}}^{\leq}) \subseteq \overline{\underline{A}_{\mathscr{R}}}(CL_{n_{2}}^{\leq}). \\ Proof:$$

1) $\forall x \notin CL_{\overline{n}}^{\geq}$, i.e. $u_{CL_{\overline{n}}}^{\geq}(x) = 0$ and $v_{CL_{\overline{n}}}^{\geq}(x) = 1$, we have

$$\begin{split} u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) &= \wedge_{y \in U} \big(u_{CL_{n}^{\geq}}(y) \vee v_{\mathscr{R}_{A}}(y,x) \big) \\ &\leq u_{CL_{n}^{\geq}}(x) \vee v_{\mathscr{R}_{A}}(x,x) \\ &= 0 = u_{CL_{n}^{\geq}}(x) \\ v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) &= \vee_{y \in U} \big(v_{CL_{n}^{\geq}}(y) \wedge u_{\mathscr{R}_{A}}(y,x) \big) \\ &\geq v_{CL_{n}^{\geq}}(x) \wedge u_{\mathscr{R}_{A}}(x,x) \\ &= 1 = v_{CL_{n}^{\geq}}(x) \end{split}$$

From discussion above, we can conclude that $A_{\mathscr{R}}(CL^{\geq}_{\overline{n}}) \subseteq CL^{\geq}_{\overline{n}}.$

On the other hand, $\forall x \in U$, if $x \in CL_{\overline{n}}^{\geq}$, i.e. $u_{CL^{\geq}_{\overline{n}}}(x) = 1$ and $v_{CL^{\geq}_{\overline{n}}}(x) = 0$, we have

$$\begin{split} u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x) &= & \lor_{y \in U} \big(u_{CL_{n}^{\geq}}(y) \wedge u_{\mathscr{R}_{A}}(x,y) \big) \\ &\geq & u_{CL_{n}^{\geq}}(x) \wedge u_{\mathscr{R}_{A}}(x,x) \\ &= & u_{CL_{n}^{\geq}}(x) \\ v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x) &= & \wedge_{y \in U} \big(v_{CL_{n}^{\geq}}(y) \vee v_{\mathscr{R}_{A}}(x,y) \big) \\ &\leq & v_{CL_{n}^{\geq}}(x) \vee v_{\mathscr{R}_{A}}(x,x) \\ &= & v_{CL_{n}^{\geq}}(x) \end{split}$$

From discussion above, we can conclude that $CL_{\overline{n}}^{\geq} \subseteq A_{\mathscr{R}}(CL_{\overline{n}}^{\geq}).$

Similarity, it is not difficult to prove $A_{\mathscr{R}}(CL_{\overline{n}}^{\leq}) \subseteq$ $CL_{\overline{n}}^{\leq} \subseteq \overline{A_{\mathscr{R}}}(CL_{\overline{n}}^{\leq}).$

2) $\forall x \in U$, since $u_{CL_n^{\geq}}(x) = v_{CL_{n-1}^{\leq}}(x)$ and $v_{CL_{n}^{\geq}}(x) = u_{CL_{n-1}^{\leq}}(x)^{n}$ where $n = 2, \cdots, m$, we have

$$\begin{aligned} u_{\underline{A}\underline{\mathscr{R}}(CL_{n}^{\geq})}(x) &= \wedge_{y \in U} \left(u_{CL_{n}^{\geq}}(y) \lor v_{\mathscr{R}_{A}}(y,x) \right) \\ &= \wedge_{y \in U} \left(v_{CL_{n-1}^{\leq}}(y) \lor v_{\mathscr{R}_{A}}(y,x) \right) \\ &= v_{\overline{A}\underline{\mathscr{R}}(CL_{n-1}^{\leq})}(x) \\ v_{\underline{A}\underline{\mathscr{R}}(CL_{n}^{\geq})}(x) &= \lor_{y \in U} \left(v_{CL_{n-1}^{\geq}}(y) \land u_{\mathscr{R}_{A}}(y,x) \right) \\ &= v_{y \in U} \left(u_{CL_{n-1}^{\leq}}(y) \land u_{\mathscr{R}_{A}}(y,x) \right) \\ &= u_{\overline{A}\underline{\mathscr{R}}(CL_{n-1}^{\leq})}(x) \end{aligned}$$

From discussion above, we can conclude that $A_{\mathscr{R}}(CL_{\overline{n}}^{\geq}) = U - \overline{A_{\mathscr{R}}}(CL_{n-1}^{\leq}), n = 2 \cdots m.$ Others can be proved analogously.

3) By Eq. 1), $\forall (x, y) \in U \times U$, we have $u_{\mathscr{R}_A}(x, y) \geq$ $u_{\mathscr{R}_{AT}}(x,y)$ and $v_{\mathscr{R}_{A}}(x,y) \leq v_{\mathscr{R}_{AT}}(x,y)$ because $A \subseteq AT$, thus

$$\begin{aligned} u_{\underline{A}\underline{\mathscr{R}}(CL_{n}^{\geq})}(x) &= & \wedge_{y \in U} \left(u_{CL_{n}^{\geq}}(y) \lor v_{\mathscr{R}_{A}}(y,x) \right) \\ &\leq & \wedge_{y \in U} \left(u_{CL_{n}^{\geq}}(y) \lor v_{\mathscr{R}_{AT}}(y,x) \right) \\ &= & u_{\underline{AT}\underline{\mathscr{R}}(CL_{n}^{\geq})}(x) \\ v_{\underline{A}\underline{\mathscr{R}}(CL_{n}^{\geq})}(x) &= & \vee_{y \in U} \left(v_{CL_{n}^{\geq}}(y) \land u_{\mathscr{R}_{A}}(y,x) \right) \\ &\geq & \vee_{y \in U} \left(v_{CL_{n}^{\geq}}(y) \land u_{\mathscr{R}_{AT}}(y,x) \right) \end{aligned}$$

$$= v_{AT_{\mathscr{R}}(CL_n^{\geq})}(x)$$

From discussion above, we can conclude that $\underline{A}_{\mathscr{R}}(CL_n^{\geq}) \subseteq \underline{AT}_{\mathscr{R}}(CL_n^{\geq})$. Others can be proved analogously.

4) Since $n_1 \leq n_2$, we obtain that $CL_{n_1}^{\geq} \supseteq CL_{n_2}^{\geq}$, i.e. $u_{CL_{n_1}^{\geq}}(x) \geq u_{CL_{n_2}^{\geq}}(x)$ and $v_{CL_{n_1}^{\geq}}(x) \leq v_{CL_{n_2}^{\geq}}(x)$ for each $x \in U$, thus

$$\begin{split} u_{\underline{A}_{\mathscr{R}}(CL_{n_{1}}^{\geq})}(x) &= \wedge_{y \in U} \left(u_{CL_{n_{1}}^{\geq}}(y) \lor v_{\mathscr{R}_{A}}(y,x) \right); \\ &\geq \wedge_{y \in U} \left(u_{CL_{n_{2}}^{\geq}}(y) \lor v_{\mathscr{R}_{A}}(y,x) \right); \\ &= u_{\underline{A}_{\mathscr{R}}(CL_{n_{2}}^{\geq})}(x) \\ v_{\underline{A}_{\mathscr{R}}(CL_{n_{1}}^{\geq})}(x) &= \vee_{y \in U} \left(v_{CL_{n_{1}}^{\geq}}(y) \land u_{\mathscr{R}_{A}}(y,x) \right) \\ &\leq \vee_{y \in U} \left(v_{CL_{n_{2}}^{\geq}}(y) \land u_{\mathscr{R}_{A}}(y,x) \right) \\ &= v_{\underline{A}_{\mathscr{R}}(CL_{n_{2}}^{\geq})}(x) \end{split}$$

From discussion above, we can conclude that $\underline{A}_{\mathscr{R}}(CL_{\overline{n_1}}^{\geq}) \supseteq \underline{A}_{\mathscr{R}}(CL_{\overline{n_2}}^{\geq})$. Others can be proved analogously.

Results 1), 2), 3) and 4) of Theorem 3 can be regarded as intuitionistic fuzzy counterparts of results well known within the classical DRSA. More precisely, 1) says that the upward (downward) union of decision classes include its intuitionistic fuzzy rough lower approximation and is included in its intuitionistic fuzzy rough upper approximation; 2) represents complementarity properties of the proposed intuitionistic fuzzy dominance–based rough approximations; 3) expresses monotonicity of the proposed intuitionistic fuzzy dominance–based rough set in terms of the monotonous varieties of condition attributes; 4) expresses monotonicity of the proposed intuitionistic fuzzy dominance–based rough set in terms of the monotonous varieties of unions of decision classes.

B. Intuitionistic fuzzy dominance–based rough set in decision system with probabilistic interpretation

It is well known that Greco's traditional DRSA was firstly proposed for dealing with complete system with preference–ordered domains of the attributes. In this section, we will illustrate how the proposed intuitionistic fuzzy dominance–based rough set can be used in the decision system with probabilistic interpretation.

For a decision system \mathscr{I} , if $\forall x \in U$ and $\forall a \in AT$, $a(x) \subseteq V_a$ instead of $a(x) \in V_a$, i.e.

$$a: U \to P(V_a)$$

where $P(V_a)$ is the collection of all nonempty subsets of V_a , then such system is referred to as a set-valued decision system. Obviously, in a set-valued decision system \mathscr{I} , x holds a set of values instead of a single value on each attribute.

Furthermore, in a set-valued decision system with probabilistic interpretation, $\forall v \in V_a$, $a(x)(v) \in [0, 1]$

represents the possibility of state v. $\forall x \in U, \forall a \in AT$, we assume here that

$$\sum_{v \in V_a} a(x)(v) = 1$$

It is clear that every set value is expressed in a probability distribution over the elements contained in such set. This leads to that the set value can be expressed in terms of a probability distribution such that

$$a(x) = \{v_1/a(x)(v_1), v_2/a(x)(v_2), \cdots, v_k/a(x)(v_k)\}$$

where $v_1, v_2, \cdots, v_k \in V_a$.

Actually, the set-valued decision system with probabilistic interpretation has been analyzed by rough set technique. For example, in valued tolerance relation [53], [54] and the valued dominance relation [60] based rough sets for dealing with incomplete information systems, each unknown value is expressed in a uniform probability distribution over the elements contained in the domain of the corresponding attribute. Suppose that $V_a =$ $\{a_1, a_2, a_3, a_4\}$, if a(x) = * where * denotes the "do not care" unknown value, then the probability distribution can be written such that

$$a(x) = \{a_1/0.25, a_2/0.25, a_3/0.25, a_4/0.25\}.$$

This tells us that if the value that x holds on a is unknown, then x may hold any one of the values in V_a . Moreover, the probabilistic degrees that x holds each value are equal. However, valued tolerance and dominance relations only consider the memberships of tolerance degree and dominance degree, they do not take the non-memberships into account. To overcome this limitation, the intuitionistic fuzzy rough technique has become a necessity.

Let us consider Table 1, it is a set-valued decision system with probabilistic interpretation. In Table 1,

- $U = \{x_1, x_2, \cdots, x_{10}\}$ is the universe of discourse;
- $AT = \{a, b, c, d, e\}$ denotes the set of condition attributes;
- $V_a = \{a_0, a_1, a_2\}, V_b = \{b_0, b_1, b_2\}, V_c = \{c_0, c_1, c_2\}, V_d = \{d_0, d_1, d_2\}, V_e = \{e_0, e_1, e_2\}, a_0 < a_1 < a_2, b_0 < b_1 < b_2, c_0 < c_1 < c_2, d_0 < d_1 < d_2, e_0 < e_1 < e_2;$
- f is the decision attribute where $V_f = \{1, 2\}$

 $\forall (x,y) \in U \times U$, let us denote the intuitionistic fuzzy dominance relation as following:

$$\mathscr{R}_{AT}(x,y) = \begin{cases} [1,0]: x = y \\ < u_{\mathscr{R}_{AT}}(x,y), v_{\mathscr{R}_{AT}}(x,y) >: \text{otherwise} \end{cases}$$

where $\forall a \in AT$,

$$u_{\mathscr{R}_{a}}(x,y) = \sum_{v_{1} > v_{2}, v_{1}, v_{2} \in V_{a}} a(x)(v_{1}) \cdot a(x)(v_{2})$$
$$v_{\mathscr{R}_{a}}(x,y) = \sum_{v_{1} < v_{2}, v_{1}, v_{2} \in V_{a}} a(x)(v_{1}) \cdot a(x)(v_{2})$$

In the above definition, $u_{\mathscr{R}_{AT}}(x, y)$ denotes the degree of *dominance principle* in terms of the set of attributes AT

U	a	b	c	d	e	f
x_1	$\{a_1/1\}$	$\{b_0/0.7, b_1/0.3\}$	$\{c_0/1\}$	$\{d_1/0/4, d_2/0.6\}$	$\{e_2/1\}$	2
x_2	$\{a_0/0.3, a_1/0.7\}$	$\{b_2/1\}$	$\{c_1/0.5, c_2/0.5\}$	$\{d_0/1\}$	$\{e_0/1\}$	1
x_3	$\{a_0/1\}$	$\{b_1/0.4, b_2/0.6\}$	$\{c_1/1\}$	$\{d_0/0.3, d_1/0.7\}$	$\{e_0/1\}$	1
x_4	$\{a_0/0.9, a_1/0.1\}$	$\{b_1/1\}$	$\{c_1/1\}$	$\{d_1/1\}$	$\{e_0/0.2, e_2/0.8\}$	1
x_5	$\{a_1/0.8, a_2/0.2\}$	$\{b_1/1\}$	$\{c_0/0.6, c_1/0.4\}$	$\{d_0/1\}$	$\{e_1/1\}$	2
x_6	$\{a_0/0.5, a_2/0.5\}$	$\{b_1/1\}$	$\{c_0/0.3, c_1/0.7\}$	$\{d_0/1\}$	$\{e_1/1\}$	1
x_7	$\{a_1/1\}$	$\{b_0/0.2, b_2/0.8\}$	$\{c_0/0.1, c_1/0.9\}$	$\{d_1/1\}$	$\{e_2/1\}$	2
x_8	$\{a_0/1\}$	$\{b_2/1\}$	$\{c_1/1\}$	$\{d_0/1\}$	$\{e_0/0.9, e_1/0.1\}$	1
x_9	$\{a_1/1\}$	$\{b_0/0.8, b_1/0.2\}$	$\{c_0/0.5, c_2/0.5\}$	$\{d_1/1\}$	$\{e_2/1\}$	2
x_{10}	$\{a_1/1\}$	$\{b_1/1\}$	$\{c_2/1\}$	$\{d_0/0.8, d_1/0.2\}$	$\{e_2/1\}$	2

 TABLE I.

 A decision system with probabilistic interpretation

 TABLE II.

 INTUITIONISTIC FUZZY DOMINANCE RELATION IN TABLE 1

$x \setminus y$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x ₁₀
x_1	[1.0,0.0]	[0.0,1.0]	[0.0,1.0]	[0.0,1.0]	[0.0,0.7]	[0.0,0.7]	[0.0,0.9]	[0.0,1.0]	[0.0,0.5]	[0.0,1.0]
x_2	[0.0, 1.0]	[1.0,0.0]	[0.0,0.7]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]	[0.0,0.1]	[0.0, 1.0]	[0.0, 1.0]
x_3	[0.0, 1.0]	[0.0,0.7]	[1.0,0.0]	[0.0,0.8]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 0.4]	[0.0, 1.0]	[0.0, 1.0]
x_4	[0.0,0.9]	[0.0, 1.0]	[0.0,0.6]	[1.0,0.0]	[0.0,0.92]	[0.0,0.5]	[0.0,0.9]	[0.0, 1.0]	[0.0,0.9]	[0.0, 1.0]
x_5	[0.0, 1.0]	[0.0, 1.0]	[0.0,0.7]	[0.0, 1.0]	[1.0,0.0]	[0.0,0.42]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]
x_6	[0.0, 1.0]	[0.0, 1.0]	[0.0,0.7]	[0.0, 1.0]	[0.0,0.5]	[1.0,0.0]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]
x_7	[0.0,0.6]	[0.0,0.55]	[0.0, 0.2]	[0.0, 0.2]	[0.0, 0.2]	[0.27,0.5]	[1.0,0.0]	[0.0, 0.2]	[0.0,0.5]	[0.0, 1.0]
x_8	[0.0, 1.0]	[0.0,0.7]	[0.0,0.7]	[0.0, 1.0]	[0.0, 1.0]	[0.0,0.9]	[0.0, 1.0]	[1.0,0.0]	[0.0, 1.0]	[0.0, 1.0]
x_9	[0.0,0.6]	[0.0, 1.0]	[0.0,0.92]	[0.0,0.8]	[0.0,0.8]	[0.0,0.8]	[0.0,0.8]	[0.0, 1.0]	[1.0,0.0]	[0.0, 0.8]
x_{10}	[0.0,0.92]	[0.0, 1.0]	[0.0,0.6]	[0.0, 0.8]	[0.0,0.2]	[0.0,0.5]	[0.0, 0.8]	[0.0, 1.0]	[0.0, 0.8]	[1.0,0.0]

while $v_{\mathscr{R}_{AT}}(x, y)$ denotes the degree of *non-dominance* principle in terms of the set of attributes AT. For instance,

$$\begin{aligned} u_{\mathscr{R}_b}(x_2, x_1) &= \sum_{\substack{v_1 \ge v_2, v_1, v_2 \in V_b}} b(x_2)(v_1) \cdot b(x_1)(v_2) \\ &= b(x_2)(b_2) \cdot b(x_1)(b_0) \\ &+ b(x_2)(b_2) \cdot b(x_1)(b_1) \\ &= 1 \\ v_{\mathscr{R}_b}(x_2, x_1) &= \sum_{\substack{v_1 < v_2, v_1, v_2 \in V_b}} b(x_2)(v_1) \cdot b(x_1)(v_2) \\ &= 0 \end{aligned}$$

Similarity, the result of intuitionistic fuzzy dominance relation in Table 1 is showed in Table 2.

By the above intuitionistic fuzzy dominance relation, we can obtain the corresponding rough approximate memberships and non-memberships by Definition 4. By the decision attribute f, the universe can be partitioned into decision classes such that $\mathbf{CL} = \{CL_1, CL_2\} =$ $\{\{x_2, x_3, x_4, x_6, x_8\}, \{x_1, x_5, x_7, x_9, x_{10}\}\}$. The results of intuitionistic fuzzy dominance-based rough approximations in Table 1 are showed in Table 3.

IV. APPROXIMATE DISTRIBUTION REDUCTS OF INTUITIONISTIC FUZZY DOMINANCE–BASED ROUGH SET

The concept of approximate distribution reduct was firstly proposed by Mi in Ref. [42]. Following Mi's work, we have introduced such reduct into DRSA for dealing with the incomplete system with lost unknown values [65]. Moreover, Qian et al. [49] also introduced such reduct into maximal consistent block based rough set approach for dealing with the incomplete system with

TABLE III. Intuitionistic fuzzy dominance–based rough approximations in Table 1

$r \setminus u$	<i>r</i> 1	ro	ro	TA.	r-	re	$r_{\overline{z}}$	r_{0}	ro	r10
$\frac{u_{\Lambda T_{\mathcal{O}}(CL^{\geq})}(x)}{u_{\Lambda T_{\mathcal{O}}(CL^{\geq})}(x)}$	1	1	1	1	1	1	$\frac{x_1}{1}$	1	1	1
$u_{AT_{\mathcal{Q}}(CL_{1})}^{\underline{AT_{\mathcal{Q}}}(CL_{1})}(x)$	0.9	0	0	0	0.5	0	0.9	0	0.9	1
$v_{AT_{\mathcal{R}}(CL^{\geq})}^{\underline{MI}_{\mathcal{R}}(CL_{2})}(x)$	0	0	0	0	0	0	0	0	0	0
$v_{AT \otimes (CL^{\geq})}^{\underline{MI} \otimes (CL_{1})}(x)$	0	1	1	1	0	1	0	1	0	0
$u \frac{\overline{AT_{\mathscr{R}}}(CL_2^2)}{\overline{AT_{\mathscr{R}}}(CL_2^2)}(x)$	1	1	1	1	1	1	1	1	1	1
$u_{\overline{AT}_{\mathscr{R}}(CL_{2}^{\geq})}(x)$	1	0	0	0	1	0	1	0	1	1
$v_{\overline{AT}_{\mathscr{R}}(CL_{1}^{\geq})}(x)$	0	0	0	0	0	0	0	0	0	0
$v_{\overline{AT_{\mathscr{R}}}(CL_{2}^{\geq})}(x)$	0	1	1	0.9	0	0.5	0	1	0	0
$u_{AT_{\mathscr{R}}(CL_{1}^{\leq})}(x)$	0	1	1	0.9	0	0.5	0	1	0	0
$u_{AT_{\mathscr{R}}(CL_{2}^{\leq})}^{\underline{1}}(x)$	1	1	1	1	1	1	1	1	1	1
$v_{AT_{\mathscr{R}}(CL_{1}^{\leq})}(x)$	1	0	0	0	1	0	1	0	1	1
$v_{AT_{\mathscr{R}}(CL_{2}^{\leq})}^{\underline{x}}(x)$	0	0	0	0	0	0	0	0	0	0
$u \frac{\overline{AT_{\mathscr{R}}}(LL_{1}^{\leq})}{\overline{AT_{\mathscr{R}}}(LL_{1}^{\leq})}(x)$	0	1	1	1	0	1	0	1	0	0
$u_{\overline{AT_{\mathscr{R}}}(CL_{2}^{\leq})}(x)$	1	1	1	1	1	1	1	1	1	1
$v_{\overline{AT_{\mathscr{R}}}(CL_{1}^{\leq})}(x)$	0.9	0	0	0	0.5	0	0.9	0	0.9	1
$v_{\overline{AT_{\mathscr{R}}}(CL_{2}^{\leq})}(x)$	0	0	0	0	0	0	0	0	0	0

"do not care" unknown values. In the following, we will further generalize the concept of approximate distribution reduct into our intuitionistic fuzzy dominance–based rough set model. A. Concept and approach to approximate distribution reducts

Definition 5: Let \mathscr{I} be a decision system, $A \subseteq AT$, let us denote by

$$\begin{split} L_{AT}^{\leq} &= \{\underline{AT_{\mathscr{R}}}(CL_{1}^{\leq}), \underline{AT_{\mathscr{R}}}(CL_{2}^{\leq}), \cdots, \underline{AT_{\mathscr{R}}}(CL_{n}^{\leq})\};\\ L_{AT}^{\leq} &= \{\underline{AT_{\mathscr{R}}}(CL_{1}^{\leq}), \underline{AT_{\mathscr{R}}}(CL_{2}^{\leq}), \cdots, \underline{AT_{\mathscr{R}}}(CL_{n}^{\leq})\};\\ H_{AT}^{\geq} &= \{\overline{AT_{\mathscr{R}}}(CL_{1}^{\geq}), \overline{AT_{\mathscr{R}}}(CL_{2}^{\geq}), \cdots, \overline{AT_{\mathscr{R}}}(CL_{n}^{\geq})\};\\ H_{AT}^{\leq} &= \{\overline{AT_{\mathscr{R}}}(CL_{1}^{\leq}), \overline{AT_{\mathscr{R}}}(CL_{2}^{\leq}), \cdots, \overline{AT_{\mathscr{R}}}(CL_{n}^{\leq})\}; \end{split}$$

- If L[≥]_A = L[≥]_{AT}, then A is referred to as the ≥-lower approximate distribution consistent set; if L[≥]_A = L[≥]_{AT} and L[≥]_B ≠ L[≥]_A for ∀B ⊂ A, then A is referred to as a ≥-lower approximate distribution reduct of 𝒴;
- 2) If $L_A^{\leq} = L_{AT}^{\leq}$, then A is referred to as the \leq -lower approximate distribution consistent set; if $L_A^{\leq} = L_{AT}^{\leq}$ and $L_B^{\leq} \neq L_A^{\leq}$ for $\forall B \subset A$, then A is referred to as a \leq -lower approximate distribution reduct of \mathscr{I} ;
- 3) If H[≥]_A = H[≥]_{AT}, then A is referred to as the ≥-upper approximate distribution consistent set; if H[≥]_A = H[≥]_{AT} and H[≥]_B ≠ H[≥]_A for ∀B ⊂ A, then A is referred to as a ≥-upper approximate distribution reduct of 𝒴;
- 4) If H[≤]_A = H[≤]_{AT}, then A is referred to as the ≤-upper approximate distribution consistent set; if H[≤]_A = H[≤]_{AT} and H[≤]_B ≠ H[≤]_A for ∀B ⊂ A, then A is referred to as a ≤-upper approximate distribution reduct of 𝒴.

A \geq -lower(upper) approximate distribution consistent set is a subset of attributes that preserves the intuitionistic fuzzy dominance-based lower(upper) approximations of all the *upward* unions of the decision classes; a \leq lower(upper) approximate distribution consistent set is a subset of attributes that preserves the intuitionistic fuzzy dominance-based lower(upper) approximations of all the downward unions of the decision classes; a \geq lower(upper) approximate distribution reduct is a minimal subset of attributes that preserves the intuitionistic fuzzy dominance-based lower(upper) approximations of all the *upward* unions of the decision classes; a \leq -lower(upper) approximate distribution reduct is a minimal subset of attributes that preserves the intuitionistic fuzzy dominancebased lower(upper) approximations of all the downward unions of the decision classes.

Theorem 4: Let \mathscr{I} be a decision system in which $A \subseteq AT$, we have

- A is ≥-lower approximate distribution consistent set ⇔ A is ≤-upper approximate distribution consistent set;
- A is ≤-lower approximate distribution consistent set ⇔ A is ≥-upper approximate distribution consistent set.

Proof: It can be derived directly from (2) of Theorem 3 and Definition 5.

Theorem 5: Let \mathscr{I} be a decision system in which $A \subseteq AT$, we have

- A is ≥-lower approximate distribution reduct ⇔ A is ≤-upper approximate distribution reduct;
- 2) A is ≤-lower approximate distribution reduct ⇔ A is ≥-upper approximate distribution reduct.
 Proof: It can be derived directly from Theorem 4

and Definition 5.

Theorem 6: Let \mathscr{I} be a decision system in which $A \subseteq AT$, for $\forall x \in U$, we denote

$$\begin{split} P_{AT}^{\geq}(x) &= \{ < u_{\underline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x), v_{\underline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x) >: n \in N \} \} \\ P_{AT}^{\leq}(x) &= \{ < u_{\underline{AT_{\mathscr{R}}}(CL_{n}^{\leq})}(x), v_{\underline{AT_{\mathscr{R}}}(CL_{n}^{\leq})}(x) >: n \in N \} \} \\ Q_{AT}^{\geq}(x) &= \{ < u_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x), v_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x) >: n \in N \} \} \\ Q_{AT}^{\leq}(x) &= \{ < u_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\leq})}(x), v_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\leq})}(x) >: n \in N \} \} \end{split}$$

then we have the following:

- A is ≥-lower approximate distribution consistent set
 ⇔ for ∀x ∈ U, P[≥]_A(x) = P[≥]_{AT}(x);
- A is ≤-lower approximate distribution consistent set
 ⇔ for ∀x ∈ U, P[≤]_A(x) = P[≤]_{AT}(x);
- A is ≥-upper approximate distribution consistent set
 ⇔ for ∀x ∈ U, Q[≥]_A(x) = Q[≥]_{AT}(x);
- 4) A is ≤-upper approximate distribution consistent set
 ⇔ for ∀x ∈ U, Q[≤]_A(x) = Q[≤]_{AT}(x).

Proof: We only prove (1), others can be proved analogously.

$$\begin{split} L_{A}^{\geq} &= L_{AT}^{\geq} \Leftrightarrow \underline{A}_{\mathscr{R}}(CL_{n}^{\geq}) = \underline{AT}_{\mathscr{R}}(CL_{n}^{\geq})(n \in \mathbb{N}) \\ \otimes u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) = u_{\underline{AT}_{\mathscr{R}}(CL_{n}^{\geq})}(x), v_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x) = v_{\underline{AT}_{\mathscr{R}}(CL_{n}^{\geq})}(x) \\ (\forall x \in U) \Leftrightarrow P_{A}^{\geq}(x) = P_{AT}^{\geq}(x). \end{split}$$

Definition 6: Let \mathscr{I} be a decision system in which $A \subseteq AT$, define

$$\begin{split} D_{L}^{\geq} &= \{(x,y) \in U^{2} : x \in U, y \notin CL_{n}^{\geq}, n = 2, \cdots, m\} \\ D_{L}^{\leq} &= \{(x,y) \in U^{2} : x \in U, y \notin CL_{n}^{\leq}, n = 1, \cdots, m-1\} \\ D_{H}^{\geq} &= \{(x,y) \in U^{2} : x \in U, y \in CL_{n}^{\geq}, n = 2, \cdots, m\} \\ D_{H}^{\leq} &= \{(x,y) \in U^{2} : x \in U, y \in CL_{n}^{\leq}, n = 1, \cdots, m-1\} \\ \text{where} \end{split}$$

- 1) if $(x,y) \in D_L^{\geq}$, then $D_L^{\geq u}(x,y) = \{a \in AT : u_{\underline{AT}_{\mathscr{R}}(CL_n^{\geq})}(x) \leq v_{\mathscr{R}_a}(y,x)\}$, otherwise, $D_L^{\geq u}(x,y) = \emptyset$;
- 2) if $(x,y) \in D_L^{\geq}$, then $D_L^{\geq v}(x,y) = \{a \in AT : v_{\underline{AT_{\mathscr{R}}}(CL_n^{\geq})}(x) \geq u_{\mathscr{R}_a}(y,x)\}$, otherwise, $D_L^{\geq v}(x,y) = \emptyset$;
- 3) if $(x,y) \in D_L^{\leq}$, then $D_L^{\leq u}(x,y) = \{a \in AT : u_{\underline{AT}_{\mathscr{R}}(CL_n^{\leq})}(x) \leq v_{\mathscr{R}_a}(x,y)\}$, otherwise, $D_L^{\leq u}(x,y) = \emptyset$;
- 4) if $(x,y) \in D_{\overline{L}}^{\leq}$, then $D_{\overline{L}}^{\leq v}(x,y) = \{a \in AT : v_{\underline{AT}_{\mathscr{R}}(CL_{n}^{\leq})}(x) \geq u_{\mathscr{R}_{a}}(x,y)\}$, otherwise, $D_{L}^{\leq v}(x,y) = \emptyset$;
- 5) if $(x,y) \in D_{H}^{\geq}$, then $D_{H}^{\geq u}(x,y) = \{a \in AT : u_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x) \geq u_{\mathscr{R}_{a}}(x,y)\}$, otherwise, $D_{H}^{\geq u}(x,y) = \emptyset;$
- 6) if $(x,y) \in D_H^{\geq}$, then $D_H^{\geq v}(x,y) = \{a \in AT : v_{\overline{AT_{\mathscr{R}}}(CL_n^{\geq})}(x) \leq v_{\mathscr{R}_a}(x,y)\}$, otherwise, $D_H^{\geq v}(x,y) = \emptyset$;

- 7) if $(x,y) \in D_H^{\geq}$, then $D_H^{\leq u}(x,y) = \{a \in$ AT : $u_{\overline{AT_{\mathscr{R}}}(CL_n^{\geq})}(x) \geq u_{\mathscr{R}_a}(y,x)$, otherwise, $D_{H}^{\leq u}(x,y) = \emptyset;$
- 8) if $(x,y) \in D_{H}^{\geq}$, then $D_{H}^{\leq v}(x,y) = \{a \in$ AT : $v_{\overline{AT}_{\mathscr{R}}(CL_{\pi}^{\geq})}(x) \leq v_{\mathscr{R}_{a}}(x,y)$, otherwise, $D_{H}^{\leq v}(x,y) = \emptyset;$

 $\begin{array}{lll} D_L^{\geq u}(x,y), & D_L^{\geq v}(x,y), & D_L^{\leq u}(x,y), & D_L^{\leq v}(x,y), \\ D_H^{\geq u}(x,y), & D_H^{\geq v}(x,y), & D_H^{\leq u}(x,y), & D_H^{\leq v}(x,y) & \text{are} \\ \text{referred to as the } \geq^u \text{-lower, } \geq^v \text{-lower, } \leq^u \text{-lower, } \leq^v \text{-} \end{array}$ lower, \geq^{u} -upper, \geq^{v} -upper, \leq^{u} -upper and \leq^{v} -upper approximate discernibility sets for pair of the objects (x, y) respectively, the matrixes

$$\begin{split} \mathbf{M}_{L}^{\geq u} &= \{D_{L}^{\geq u}(x,y) : (x,y) \in D_{L}^{\geq}\},\\ \mathbf{M}_{L}^{\geq v} &= \{D_{L}^{\geq v}(x,y) : (x,y) \in D_{L}^{\geq}\},\\ \mathbf{M}_{L}^{\leq u} &= \{D_{L}^{\leq u}(x,y) : (x,y) \in D_{L}^{\leq}\},\\ \mathbf{M}_{L}^{\leq v} &= \{D_{L}^{\leq v}(x,y) : (x,y) \in D_{L}^{\leq}\},\\ \mathbf{M}_{H}^{\geq u} &= \{D_{H}^{\geq u}(x,y) : (x,y) \in D_{H}^{\geq}\},\\ \mathbf{M}_{H}^{\geq u} &= \{D_{H}^{\geq v}(x,y) : (x,y) \in D_{H}^{\geq}\},\\ \mathbf{M}_{H}^{\leq u} &= \{D_{H}^{\leq u}(x,y) : (x,y) \in D_{H}^{\leq}\},\\ \mathbf{M}_{H}^{\leq u} &= \{D_{H}^{\leq u}(x,y) : (x,y) \in D_{H}^{\leq}\},\\ \mathbf{M}_{H}^{\leq u} &= \{D_{H}^{\leq u}(x,y) : (x,y) \in D_{H}^{\leq}\},\\ \mathbf{M}_{H}^{\leq v} &= \{D_{H}^{\leq v}(x,y) : (x,y) \in D_{H}^{\leq}\}, \end{split}$$

are referred to as \geq^{u} -lower, \geq^{v} -lower, \leq^{u} -lower, \leq^{v} lower, \geq^{u} -upper, \geq^{v} -upper, \leq^{u} -upper and \leq^{v} -upper approximate distribution discernibility matrixes respectively.

Theorem 7: Let \mathscr{I} be a decision system in which $A \subseteq$ AT, we have

- 1) $u_{\underline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x) = u_{\underline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_L^{\geq u}(x, y) \neq \emptyset$ for each $(x, y) \in D_L^{\geq}$;
- 2) $v_{AT_{\mathscr{R}}(CL_{n}^{\geq})}(x) = v_{A_{\mathscr{R}}(CL_{n}^{\geq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_L^{\geq v}(x, y) \neq \emptyset$ for each $(x,y) \in D_L^{\leq};$
- 3) $u_{\underline{AT}_{\mathscr{R}}(CL_{n}^{\leq})}(x) = u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\leq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_L^{\leq u}(x,y) \neq \emptyset$ for each $(x, y) \in D_L^{\leq}$;
- 4) $v_{AT_{\mathscr{R}}(CL_{n}^{\leq})}(x) = v_{A_{\mathscr{R}}(CL_{n}^{\leq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_L^{\leq v}(x,y) \neq \emptyset$ for each $(x,y) \in D_L^{\leq};$
- 5) $u_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x) = u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_{H}^{\geq u}(x, y) \neq \emptyset$ for each $(x, y) \in D_H^{\leq}$;
- 6) $v_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x) = v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_H^{\geq v}(x, y) \neq \emptyset$ for each $(x,y) \in D_H^{\geq};$
- 7) $u_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\leq})}(x) = u_{\overline{A_{\mathscr{R}}}(CL_{n}^{\leq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_H^{\leq u}(x, y) \neq \emptyset$ for each $(x, y) \in D_{\overline{H}}^{\leq};$
- 8) $v_{\overline{AT_{\mathscr{R}}}(CL_{n}^{\leq})}(x) = v_{\overline{A_{\mathscr{R}}}(CL_{n}^{\leq})}(x)$ for each $x \in U$ and $n \in N$ if and only if $A \cap D_H^{\leq v}(x,y) \neq \emptyset$ for each $(x,y) \in D_H^{\geq}$.

Proof: We only prove (1), others can be proved analogously.

If n = 1, then $u_{AT_{\mathscr{R}}(CL_{1}^{\geq})}(x) = u_{A_{\mathscr{R}}(CL_{1}^{\geq})}(x) = 1$ because $CL_1^{\geq} = U$. What should be considered in the following are n > 1.

 \Rightarrow : Suppose $\exists (x, y) \in D_L^{\geq}$ such that $A \cap D_L^{\geq u}(x, y) =$ \emptyset , then for each $a \in A$, we have $u_{AT_{\mathscr{R}}(CL^{\geq}_{n})}(x) > 0$ $v_{\mathcal{R}_{a}}(y,x)$ by Definition 6. By formula (1) we have $v_{\mathscr{R}_A}(y,x) = \bigvee \{ v_{\mathscr{R}_a}(y,x) : a \in A \},$ from which we can conclude that $u_{AT_{\mathscr{R}}(CL_{n}^{\geq})}(x) > v_{\mathscr{R}_{A}}(y,x)$. Since by assumption we have $u_{\underline{AT}_{\mathscr{R}}(CL_{n}^{\geq})}(x) = u_{\underline{A}_{\mathscr{R}}(CL_{n}^{\geq})}(x),$ thus $u_{A_{\mathscr{R}}(CL^{\geq}_{n})}(x) > v_{\mathscr{R}}(\overline{y,x})$ holds, which contradictive to the condition $u_{A_{\mathscr{R}}(CL_n^{\geq})}(x) \leq v_{\mathscr{R}_A}(y,x)$ because $u_{A_{\mathscr{R}}(CL_{\overline{n}}^{\geq})}(x) \ = \ \wedge \{\overline{v_{\mathscr{R}_{A}}}(y,x) \ : \ y \ \notin \ CL_{\overline{n}}^{\geq}\} \ (n \ = \ n \)$ $2\overline{,\cdot\cdot\cdot},m$).

 \Leftarrow : Suppose that $\exists x \in U$ and $n \in N$ where $u_{\underline{AT_{\mathscr{R}}}(CL_{n}^{\geq})}(x) \neq u_{\underline{A_{\mathscr{R}}}(CL_{n}^{\geq})}(x), \text{ then } n = 2, \cdots, m$ $\mathrm{and} \ \overline{u}_{AT_{\mathscr{R}}(CL_{n}^{\geq})}(x) \ \overline{>} \ u_{A_{\mathscr{R}}(CL_{n}^{\geq})}(x) \ \text{(by Theorem 3)}.$ Therefore, there must be $y \notin CL_n^{\geq}$ such that $v_{\mathscr{R}_A}(y, x) <$ $u_{\underline{AT}_{\mathscr{R}}(CL_{n}^{\geq})}(x)$, it follows that for each $a \in A$, $v_{\mathscr{R}_a}(y,x) < u_{\underline{AT_{\mathscr{R}}}(CL_n^{\geq})}(x)$ holds, i.e. $A \cap D_L^{\geq u}(x,y) =$ \emptyset , here $(x, y) \in D_L^{\geq}$. From discussion above, we can draw the following conclusion: if for each $(x, y) \in D_L^{\geq}$ where $A\cap D_L^{\geq u}(x,y)\neq \emptyset \text{, then } u_{\underline{AT_{\mathscr{R}}(CL_n^{\geq})}}(x)=u_{\underline{A_{\mathscr{R}}(CL_n^{\geq})}}(x)$ for each $x \in U$ and $n = 2, \dots, m$.

Theorem 8: Let \mathscr{I} be a decision system in which $A \subseteq$ AT, we have

- 1) $L_{A}^{\geq} = L_{AT}^{\geq} \Leftrightarrow \forall (x, y) \in D_{L}^{\geq}$ such that $A \cap (D_{L}^{\geq u}(x, y) \cap D_{L}^{\geq v}(x, y)) \neq \emptyset$; 2) $L_{A}^{\leq} = L_{AT}^{\leq} \Leftrightarrow \forall (x, y) \in D_{L}^{\leq}$ such that $A \cap (D_{L}^{\leq u}(x, y) \cap D_{L}^{\leq v}(x, y)) \neq \emptyset$;
- 3) $H_{A}^{\geq} = H_{AT}^{\geq} \Leftrightarrow \forall (x,y) \in D_{H}^{\geq}$ such that $A \cap (D_{H}^{\geq u}(x,y) \cap D_{H}^{\geq v}(x,y)) \neq \emptyset;$
- 4) $H_{A_{-}}^{\leq} = H_{AT}^{\leq} \Leftrightarrow \forall (x,y) \in D_{H}^{\leq}$ such that $A \cap$ $(D_H^{\leq u}(x,y) \cap D_H^{\leq v}(x,y)) \neq \emptyset.$

Proof: We only prove (1), others can be proved analogously.

 \Rightarrow : If $L_A^{\geq} = L_{AT}^{\geq}$, then $\forall n \in N$ and $\forall x \in U$, we have $u_{AT_{\mathscr{R}}(CL_n^{\geq})}(x) = u_{A_{\mathscr{R}}(CL_n^{\geq})}(x) \text{ and } v_{AT_{\mathscr{R}}(CL_n^{\geq})}(x) =$ $v_{A_{\mathscr{R}}(CL_n^{\geq})}(x).$ By Theorem 7, we have $A\overline{\cap D_L^{\geq u}}(x,y)\neq \emptyset$ and $A \cap D_L^{\geq v}(x,y) \neq \emptyset$ for each $(x,y) \in D_L^{\geq}$, it follows that $A \cap (D_L^{\geq u}(x, y) \cap D_L^{\geq v}(x, y)) \neq \emptyset$.

 $\Leftarrow: \text{ If } A \cap (D_L^{\geq u}(x,y) \cap D_L^{\geq v}(x,y)) \neq \emptyset \text{ for each }$ $(x,y) \in D_L^{\geq}$, then we have $A \cap D_L^{\geq u}(x,y) \neq \emptyset$ and $A \cap D_L^{\geq v}(x,y) \neq \emptyset$. By Theorem 7, we have $u_{AT_{\mathscr{R}}(CL_n^{\geq})}(x) \ = \ u_{A_{\mathscr{R}}(CL_n^{\geq})}(x) \ \text{ and } \ v_{AT_{\mathscr{R}}(CL_n^{\geq})}(x) \ =$ $v_{\underline{A}_{\underline{\mathscr{R}}}(CL_{n}^{\geq})}(x)$ for each $n \in N$ and $x \in U$, it follows that $L_{\overline{A}}^{\geq} = L_{\overline{AT}}^{\geq}$.

Definition 7: Let I be a decision system, define

$$\Delta_{L}^{\geq} = \bigwedge_{(x,y)\in D_{L}^{\geq}} ((\bigvee D_{L}^{\geq u}(x,y)) \bigwedge (\bigvee D_{L}^{\geq v}(x,y))); \quad (2)$$

$$\Delta_{L}^{\leq} = \bigwedge ((\bigvee D_{L}^{\leq u}(x,y)) \bigwedge (\bigvee D_{L}^{\leq v}(x,y))); \quad (3)$$

 $_{(x,y)\in D_{L}^{\leq}}$

$$\Delta_{H}^{\geq} = \bigwedge_{(x,y)\in D_{H}^{\geq}} ((\bigvee D_{H}^{\geq u}(x,y)) \bigwedge (\bigvee D_{H}^{\geq v}(x,y))); \quad (4)$$

$$\Delta_{H}^{\leq} = \bigwedge_{(x,y)\in D_{H}^{\leq}} ((\bigvee D_{H}^{\leq u}(x,y)) \bigwedge (\bigvee D_{H}^{\leq v}(x,y))); \quad (5)$$

 $\Delta_L^{\geq}, \Delta_L^{\leq}, \Delta_H^{\geq}$ and Δ_H^{\leq} are referred to as the \geq -lower, \leq -lower, \geq -upper and \leq -upper approximate discernibility functions respectively.

By using Boolean reasoning techniques, we can obtain the following Theorem 9 from Theorem 8.

Theorem 9: Let \mathscr{I} be a decision system in which $A \subseteq AT$, then we have

- 1) A is \geq -lower approximate distribution reduct $\Leftrightarrow \bigwedge A$ is a prime implicant of Δ_L^{\geq} ;
- 2) A is \leq -lower approximate distribution reduct $\Leftrightarrow \bigwedge A$ is a prime implicant of Δ_L^{\leq} ;
- 3) A is \geq -upper approximate distribution reduct $\Leftrightarrow \bigwedge A$ is a prime implicant of Δ_H^{\geq} ;
- 4) A is \leq -upper approximate distribution reduct $\Leftrightarrow \bigwedge A$ is a prime implicant of Δ_H^{\leq} .

Proof: We only prove (1), others can be proved analogously.

" \Rightarrow ": Since A is \geq -lower approximate distribution reduct, then A is also a \geq -lower approximate distribution consistent set. By Theorem 8, we have $A \cap (D_L^{\geq u}(x,y) \cap D_L^{\geq v}(x,y)) \neq \emptyset, \forall (x,y) \in D_L^{\geq}$. We claim that for each $a \in A$, there must be $(x,y) \in D_L^{\geq}$ such that $A \cap (D_L^{\geq u}(x,y) \cap D_L^{\geq v}(x,y)) = \{a\}$. In fact, if for each pair $(x,y) \in D_L^{\geq}$, there exists $a \in D_L^{\geq}(x,y)$ such that $Card(A \cap (D_L^{\geq u}(x,y) \cap D_L^{\geq v}(x,y))) > 2$ where $a \in A \cap (D_L^{\geq u}(x,y) \cap D_L^{\geq v}(x,y))$, let $A' = A - \{a\}$, then by Theorem 8 we can see that A' is a \geq -lower approximate distribution consistent set, which contradicts that A is a \geq -lower approximate distribution reduct . It follows that $\bigwedge A$ is a prime implicant of Δ_L^{\geq} .

"⇐": If $\bigwedge A$ is a prime implicant of $\Delta_{\overline{L}}^{\geq}$, then by Theorem 8 there must be $A \cap (D_L^{\geq u}(x,y) \cap D_L^{\geq v}(x,y)) \neq \emptyset$, $(\forall (x,y) \in D_L^{\geq})$. Moreover, for each $a \in A$, there exists $(x,y) \in D_{\overline{L}}^{\geq}$ such that $A \cap (D_{\overline{L}}^{\geq u}(x,y) \cap D_{\overline{L}}^{\geq v}(x,y)) = \{a\}$. Consequently, $\forall A'$ where $A' \subseteq A$ and $A' = A - \{a\}$, A' is not the \geq -lower approximate distribution consistent set. We conclude that A is a \geq -lower approximate distribution reduct.

B. Illustrative example

Following Section 3.2, compute the \geq -lower approximate distribution reduct, \leq -lower approximate distribution reduct and \leq -upper approximate distribution reduct of Table 1.

By Definition 6, we can obtain eight different types of distribution discernibility matrixes. Here, we only present \geq^{u} -lower, \geq^{v} -lower, \geq^{u} -upper, \geq^{v} -upper approximate distribution discernibility matrixes matrixes as Table 4, Table 5, Table 6 and Table 7 show respectively.

Therefore, by Definition 7, we obtain the following \geq -

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TABLE IV. \geq^{u} -lower approximate distribution discernibility matrix

$D_L^{\geq}(x,y)x_1$	x_2	x_3	$x_4 x$	$x_5 x_6 x_6$	$7 x_8$	$x_9 x_{10}$
x_1	$\{d, e\}$	$\{a, e\}$	$\{a\}$	$\{d, e\}$	$\{a, d, e\}$	
x_2	AT	AT	AT	AT	AT	
x_3	AT	AT	AT	AT	AT	
x_4	AT	AT	AT	AT	AT	
x_5	$\{e\}$	$\{a, e\}$	$\{a\}$	$\{a\}$	$\{a, e\}$	
x_6	AT	AT	AT	AT	AT	
x_7	$\{d, e\}$	$\{a, e\}$	$\{a\}$	$\{d, e\}$	$\{a, d, e\}$	
x_8	AT	AT	AT	AT	AT	
x_9	$\{d, e\}$	$\{a, e\}$	$\{a\}$	$\{d, e\}$	$\{a, d, e\}$	
x_{10}	$\{e\}$	$\{a, c, e\}$	$\{c\}$	$\{c, e\}$	$\{a, c, e\}$	

TABLE V. \geq^{v} -lower approximate distribution discernibility matrix

$D_L^{\geq}(x,y)x_1$	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$x_9 x_{10}$
x_1	$\{a, d, e\}$	$\{a, d, e\}$	$\{a, d, e\}$		$\{d, e\}$		$\{a, d, e\}$	
x_2	AT	AT	AT		AT		AT	
x_3	AT	AT	AT		AT		AT	
x_4	AT	AT	AT		AT		AT	
x_5	$\{a, d, e\}$	$\{a, e\}$	$\{a,b\}$		$\{b, d, e\}$		$\{a, d, e\}$	
x_6	AT	AT	AT		AT		AT	
x_7	$\{a, d, e\}$	$\{a, d, e\}$	$\{a, d, e\}$		$\{d, e\}$		$\{a, d, e\}$	
x_8	AT	AT	AT		AT		AT	
x_9	$\{a, d, e\}$	$\{a, d, e\}$	$\{a, d, e\}$		$\{d, e\}$		$\{a, d, e\}$	
x_{10}	$\{a, c, d, e\}$	$\{a, c, e\}$	$\{a, b, c, e\}$	} .	$\{b, c, d, e\}$	} {	$\{a, c, d, e\}$	

TABLE VI. \geq^u -upper approximate distribution discernibility matrix

D_H^{\geq}	(x,y)	x_1	$x_2 x_3 x_4$	x_5	x_6	x_7	x_8	x_9	x_{10}
5	r_1	AT		AT		AT		AT	AT
5	r_2	$\{a, d, e\}$	}	$\{a, d, e\}$	} {	a, d, e	} {	a, d, e	$\{a, c, d, e\}$
5	r_3	$\{a, d, e\}$	}	$\{a, e\}$	{	a, d, e	} {	a, d, e	$\{a, c, e\}$
5	r_4	$\{a, d, e\}$	}	$\{a, b\}$	{	a, d, e	} {	a, d, e	$\{a, b, c, e\}$
5	r_5	AT		AT		AT		AT	AT
5	r_6	$\{d, e\}$		$\{b, d, e\}$	} .	$\{d, e\}$		$\{d, e\}$	$\{b, c, d, e\}$
5	c_7	AT		AT		AT		AT	AT
5	r_8	$\{a, d, e\}$	}	$\{a, d, e\}$	} {	a, d, e	} {	a, d, e	$\{a, c, d, e\}$
5	r_9	AT		AT		AT		AT	AT
x	10	AT		AT		AT		AT	AT

TABLE VII. \geq^v -upper approximate distribution discernibility matrix

$D_H^{\geq}(x,y)$	$) x_1$	$x_2 x_3 x_4$	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	AT		AT		AT		AT	AT
x_2	$\{d, e\}$		$\{e\}$		$\{d, e\}$		$\{d, e\}$	$\{e\}$
x_3	$\{a, e\}$	1	[a, e]	ł	$\{a, e\}$		$\{a, e\}$	$\{a, c, e\}$
x_4	$\{a\}$		$\{a\}$		$\{a\}$		$\{a\}$	$\{a, c\}$
x_5	AT		AT		AT		AT	AT
x_6	$\{a, d, e\}$	}	$\{a\}$	{	a, b, d, e	} {	a,c,d,e	$\{a, c, e\}$
x_7	AT		AT		AT		AT	AT
x_8	$\{a, d, e\}$	}	$\{a\}$		$\{a, d, e\}$		$\{a, d, e\}$	$\{a, c, e\}$
x_9	AT		AT		AT		AT	AT
x_{10}	AT		AT		AT		AT	AT

lower, \geq -upper approximate discernibility functions:

$$\begin{array}{lll} \Delta_L^{\geq} &=& \bigwedge_{(x,y)\in D_L^{\geq}} ((\bigvee D_L^{\geq u}(x,y))\bigwedge (\bigvee D_L^{\geq v}(x,y)));\\ &=& a\bigwedge c\bigwedge e\\ \Delta_H^{\geq} &=& \bigwedge_{(x,y)\in D_H^{\geq}} ((\bigvee D_H^{\geq u}(x,y))\bigwedge (\bigvee D_H^{\geq v}(x,y)))\\ &=& a\bigwedge e \end{array}$$

By the above results and Theorem 9, we know that $\{a, c, e\}$ is the \geq -lower approximate distribution reduct of Table 1, $\{a, e\}$ is the \geq -upper approximate distribution reduct of Table 1. In other words, to preserve the intuitionistic fuzzy dominance-based lower approximations of all the upward unions of the decision classes, attributes b and d can be deleted; to preserve the intuitionistic fuzzy dominance-based upper approximations of all the upward unions of the decision classes, attributes b, c, d are redundant.

Similar to the above progress, it is not difficult to obtain that $\{a, e\}$ is the \leq -lower approximate distribution reduct of Table 1, $\{a, c, e\}$ is the \leq -upper approximate distribution reduct of Table 1. Such results demonstrate the correctness of Theorem 5.

V. CONCLUSIONS

In this paper, we have developed a general framework for the generalization of dominance–based rough set. In our approach, the concept of intuitionistic fuzzy set is combined with the DRSA and then the intuitionistic fuzzy dominance–based rough set is defined. We also introduced the concept of approximate distribution reducts into intuitionistic fuzzy dominance–based rough set model, four types of approximate distribution reducts are presented, the practical approaches to compute these reducts are also discussed. Different from the previous DRSA, we use an intuitionistic fuzzy dominance relation instead of the crisp or fuzzy dominance relation to defined dominance–based rough set model.

Furthermore, a lot of experiment analysis are also needed to conduct in the future for practical applications of our intuitionistic fuzzy dominance–based rough set approach.

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