Compositional Verification of Liveness Property in Inhibitor-arc Connections of Petri Net Systems

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Abstract—**Petri net systems synthesis can construct large systems without the requirement of reachability analysis so that it can reduce the high complexity of analyzing global system. In a synthesis process, such good properties of subsystems as liveness and deadlock-freeness etc, must be preserved in synthesized system. This paper focusses on liveness preservation in inhibitor-arc connection operations. The systems dynamic, concurrent behavior relation i.e. concurrent language relation in inhibitor-arc connections is stressed studied. The corresponding language relation formula is present and proved, and it can be applied to determine liveness of synthesized system in inhibitor-arc connection operations. Furthermore, some criteria are introduced, which are necessary and sufficient for liveness, to determine the liveness of global system by the same ones of local systems. Finally, some examples are given, illustrating the effectiveness of the proposed approach in modeling and analyzing of large systems.**

Index Terms—**liveness preservation, concurrent language, dynamic invariance, inhibitor-arc connections**

I. Introduction

Compositional methods take advantages of the modular structure of the model to build manageable state space indicating the global behavior. There are many proposals for composing Petri nets and for splitting large models into smaller ones recently. In composition, basically, any two or more nets can be composed by one or more composition operators, which forms a new larger net, which, in principle, can be very different from the original nets.

Petri net systems composition can alleviate state space exploration by guaranteeing such good properties as liveness, deadlock-freeness, bounedness, reversibility and so forth while incrementally expanding the subsystems. Thus, composition operations are an effective way to manipulate industrial size systems, and are playing an increasingly important role in theoretical and industrial fields. Normally, composition operations should obey the following three principles:

1. Preservation: The synthesized system should preserve some good properties.
2. Simplicity: The synthesis rules must be as simple as possible.
3. Generality: The rules should be as powerful as possible to generate as many classes of system as possible.

A lot of efforts has been done in this area. Wolfgang Reisig[1] provided the formal framework for a simple composition operator, adequate for many classes of Petri net applications. It requires a minimum of fairly intuitive technicalities from its users and readers. The operator furthermore is associative, thus meeting the minimal algebraic requirements when composing a large system out of several smaller ones. A Petri net model is introduced in [2],[3] which defines a set of basic subnets, namely elementary control tasks (ECT). Such a model can be applied to design logic controllers by bottom-up approach, and the subnets are used to model subsystems through a number of connection operations including self-loops, inhibitor-arc, and synchronization. The liveness preservation of Petri net in above operations of ECT_s are discussed. The work of H.Q.Wang[4] studied system behavior, and investigate system behavior in the synthesis of Petri net models by using operation of self-loops, inhibitors as well as synchronization. The approaches in [4] are only based on sequential language not on concurrent language so that their results merely suit for sequential language.

C.J.Jiang[5,19]proposed the property of dynamic invariance such as state invariance and behavior invariance in synchronous and sharing operations. They presented and proved a language relation formula in some different synthesis of Petri net systems such as well-, under- and overmatched refined Petri nets, and furthermore discussed behavior characteristics and property preservation in these compositions of subsystems. The same is that their approach is only based on sequential language not on concurrent language.

Y.Souissi[6] discussed liveness preservation in sharing synthesis of Petri net systems and proposed the concept of F-strong net on the basis of generalization of the

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monotonicity of liveness. However, there is a conjecture, which is still not solved, that whether the resultant system is a F-strong net. Y.Souissi[7] also studied the liveness preservation in other synthesis ways of Petri net systems.

M.D.Jeng[8]proposed synthesis technique which allow to model flexible manufacturing systems(FMS) guaranteeing property of liveness. In his work, each subsystem is modeled as a resource control net module, and the net system is obtained by merging the modules conforming to two minimal restrictions and the system's structural liveness is checked by an algorithm. [9] described the synthesis approaches for modeling FMS, these approaches can be guaranteeing the conservativeness of the synchronized net.

A.Aybar and A.Ifar[10] considered overlapping decompositions and expansions in order to design decentralized controllers for discrete-event systems (DESs) modeled by Petri nets. It is shown that properties like boundedness, reversibility, and liveness (with a mild additional condition) carry over from the including net to the included net.

J.Esparza and M.Silva[11] considered the synthesis of live and bounded free choice(LBFC) systems. They showed that the class of LBFC systems can be reduced/top-down synthesized by means of kits of two rules, and furthermore, the class of LBFC systems can be synthesized by means of modular compositions. On the other hand, they presented the exact conditions for the preservation of liveness and boundedness under synchronization of nets.

The work of J.Esparza[12]and [13] proposed two rules to synthesized live and bounded free-choice Petri nets. In their work, Rule RF_1 refined a macroplace by means of a state machine, and RF_2 added a marking structurally implicit place to a free-choice net. With RF_1 and RF_2, they can synthesized all live and bounded free-choice nets starting from a circuit containing only one place and one transition. However, one needs to decide whether the net can be reduced to a macroplace(in RF_1) and whether a place is an implicit place (in RF_2), and their rules are non-local ones. That is, global properties of the net must be checked in order to know if the rules are applicable. Although [12] added the RF_3 rule to synthesize expanded free-choice Petri nets, [12] was unable to synthesize asymmetric free-choice nets.

The modular state space technique[14] takes advantage of the modular organization of the model. Modular Petri nets consist only of modules synchronized through shared transitions, i.e. synchronous synthesis of Petri nets. This modular approach can often decrease the complexity of the analysis task.

Since few works have been done on liveness preservation using language approaches in four main kinds of synthesis operations, namely, sharing, synchronous, self-loops and inhibitor-arc operations, and the conventional Petri net synthesis approaches, in general, suffer the drawback of only being able to synthesize a few classes of nets such as free choice nets, it is appealing to relax these constraints in application of modeling complex systems.

Petri net system with inhibitor arcs has the possibility of testing whether a place is empty in the current marking (zero testing). Thus, they are very well suited to model situation involving testing for a specific condition. Hence, the inhibitor-arc connection is one of most significant synthesis operations. We in this paper place the emphasis on concurrent language relation in inhibitor-arc connection operation, and apply it to judge the liveness of resultant system.

This paper is structured as follows: Section 2 introduces some basic concepts and notations of Petri net system. In section 3, we present a concurrent language relation formula in inhibitor-arc connection operation. Section 4 discusses the property of dynamic invariance in inhibitor-arc connection operation. Section 5 discusses how to judge liveness of synthesized system which is composed by local subsystems in inhibitor-arc connection. Section 6 proposes some necessary and sufficient conditions to preserve liveness in inhibitor-arc connection process. Section 7 gives some concluding remarks.

II. Basic definition and notations

A Petri net is a triple $N = (P, T; F)$ such that P and T are disjoint finite sets, $F \subseteq (P \times T) \cup (T \times P)$ and $dom(F) \cup cod(F) = P \cup T$. The elements of P and T are respectively called places and transitions, and F is called the flow relation. The pair (N, M_0) is a Petri net system, where M_0 is the initial marking.

An inhibitor Petri net system is a net system with a set of inhibitor leading from places to transitions(in diagrams, inhibitor arcs have small circles as arrowheads). Given an inhibitor net system $PN = (P, T; F, I, M_0)$, where $I \subseteq P \times T$ is the set of inhibitor arcs, and $x \in P \cup T$, the preset of x denoted by ${}^\bullet x$, is defined by ${}^\bullet x = \{y | y \in P \cup T, (y, x) \in F\}$, the postset of x, denoted by $x^\bullet$, is defined by $x^\bullet = \{y | y \in P \cup T, (x, y) \in F\}$. In addition, for all $t \in T$, ${}^\circ t = \{p \in P | (p, t) \in I\}$ denotes the set of inhibiting places of t. Transition $t \in T$ is enabled at a marking M if for all $p \in^\bullet t \backslash {}^\circ t$: $M(p) > 0$ and for $p \in^\circ t$, $M(p) = 0$ where $\backslash$ is the subtraction of sets.

Transitions represents actions which may occur at a given marking and then lead to a new marking. Here, we define this dynamics in the more general terms of a set(concurrently occurring) of transitions. A step $R_t \in 2^{T^*}$ is a set of transitions which can occur concurrently, where T^* is the closure of T and 2^{T^*} denotes the power set of T^*. For example, let $R_t = \{a, b\}$, it means that transitions a and b can occur simultaneously. We also represent R_t as $\begin{pmatrix} a \\ b \end{pmatrix}$.

Definition 1: Let $PN = (P, T; F, M_0)$ be a Petri net system, $L_S(PN) = \{\alpha | \alpha \in T^* \; and \; M_0[\alpha >\}$, $L_S(PN)$ is called the sequential language of PN or the sequential behavior of PN. $L_C(PN) = \{\alpha | \alpha \in (2^{T^*})^* \; and \; M_0[\alpha >\}$, $L_C(PN)$ is called the concurrent language of PN or the concurrent behavior of PN. Usually, for $R_t \in 2^{T^*}$, if $|R_t| = 0$, R_t is called an empty step and denoted by $R_t = \lambda$. If $|R_t| = 1$, R_t is called a single step. If $|R_t| > 1$, R_t is called a concurrent step.

***Definition** 2:* Let L be a concurrent language of PN. α, β are step sequences of L. We define two operations "$\circ$" and "$+$" as following:

$\alpha \circ \beta \equiv \alpha\beta \ and \ \alpha + \beta \equiv \{\alpha, \beta\}$

Then the operation "$\circ$" and "$+$" are called connection and addition operations respectively.

***Definition** 3:* Let $PN = (N, M_0)$ be a Petri net system, where $N = (P, T; F)$. PN is live iff (if and only if) $\forall M \in [M_0 >, \forall t \in T, \exists M' \in [M >: M'[t >$. PN is deadlock-free iff $\forall M \in [M_0 >, \exists t \in T : M[t >$.

***Definition** 4:* [15, 16, 17] Let $PN_i = (P_i, T_i; F_i, M_{0_i})$(i=1,2) be Petri net systems, $P_1 \cap P_2 = \emptyset$ and $T_1 \cap T_2 = \emptyset$. Let $PN = (P, T; F, M_0)$ such that 1) $P = P_1 \cup P_2$; 2) $T = T_1 \cup T_2$; 3) $F = F_1 \cup F_2 \cup \{(p^i, t^{3-i}) | p^i \in P_i, t^{3-i} \in T_{3-i}$ and (p^i, t^{3-i}) is an inhibitor arc $\}$(it means that an inhibitor arc (p^i, t^{3-i}) always leads from a place of a Petri net PN_1 to a transition of another Petri net PN_2 if *i*=1); 4) $M_0(p) = M_{0_i}(p)$, if $p \in P_i$(i=1,2). Then PN is called an inhibitor-arc connection net system of PN_1 and PN_2, denoted as $PN = PN_1 O_I PN_2$.

$T_0 = \{t^i | \exists p^{3-i} \in P_{3-i}$ such that $(p^{3-i}, t^i) \in F$ and (p^{3-i}, t^i) is an inhibitor arc(i=1,2)$\}$ is called the set of zero-one transitions(ZOT) of T, and let $T_{0_i} = T_0 \cap T_i$, if $t \in T_{0_i}$, t is called a ZOT of T_i as well.

Remark 1: Consider example 1, PN is synthesized by PN_1 and PN_2 through two inhibitor arcs $(p5, a)$ and $(p3, f)$. From Definition 1, it is easy to know that $eafb \in L_S(PN)$ (In PN, transitions e, a, f and b occur in sequence), $ea\begin{pmatrix} b \\ f \end{pmatrix} \in L_C(PN)$ (In PN, transitions e, a occur in sequence and b, f occur concurrently). For step sequence $ea\begin{pmatrix} b \\ f \end{pmatrix}$, there are three steps, i.e., two single steps $R_{t_1} = \{e\}$, $R_{t_2} = \{a\}$ and one concurrent step $R_{t_3} = \begin{pmatrix} b \\ f \end{pmatrix}$. Transitions a and f are ZOT_s of T where a is a ZOT of T_1 and f is a ZOT of T_2 respectively.

***Definition** 5:* Let $PN = (P, T; F, M_0)$ be a Petri net system. 1)$\forall P' \subseteq P$, $\forall M \in [M_0 >$, denote $M|_{P'}$ as the restriction of M on P'. i.e. $M|_{P'}(p) = M(p), \forall p \in P'$. 2) $\forall T' \subseteq T$, $\alpha \in (2^{T^*})^*$, $\alpha|_{T'}$ is the sequences got from α by removing all occurrence of transitions not belonging to T'.

***Definition** 6:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$(i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$.

1) A step sequence $\alpha \in (2^{T^*})^* (or\ \alpha \in (2^{T_i^*})^*)$ is called a path on $PN (or\ PN_i)$, if $\exists M, M' \in [M_0 >$ such that $M[\alpha > M'$.
2) For an arbitrary path $\alpha_1 \in (2^{T_i^*})^*$ on PN_i with $\alpha_1 \cap 2^{T_{0_i}^*} = \emptyset$, written as $M_1[\alpha_1 > M_2$, where $M_1 \in [M_0 >$, if there exists a path $\beta \in (2^{T_{3-i}^*})^*$ with $\beta \cap 2^{T_{0_{3-i}}^*} = \emptyset$ such that $M_2[\beta > M_3$ and $\exists R_t \in 2^{T_{0_i}^*}$ satisfying $M_3[R_t >$, then $\alpha = \alpha_1 \circ R_t$ is called a basic zero-one path($BZOP$) on PN_i, β is called a mutual zero-one path ($MZOP$) on PN_{3-i}, and α, β are called a pair of $MZOP_s$. Denote R_t as (α, β), then α can be rewritten as $\alpha = \alpha_1 \circ (\alpha, \beta)$. (α_1 and β may might be λ.)
3) Let α be a path on PN_i satisfying $\alpha \cap 2^{T_{0_i}^*} = \emptyset$, α is called a basic non-zero-one path($BNZOP$) on PN_i.

Remark 2: For instance, according to Figure 1, $\alpha = \alpha_1 \circ (\alpha, \beta)$ is a $BZOP$ on PN_2 and β is a $MZOP$ on PN_1 where $\alpha_1 = e$, $\beta = \lambda$ and $(\alpha, \beta) = f$. We can also get that $\alpha' = \alpha_1' \circ (\alpha', \beta')$ is a $BZOP$ on PN_1 and β' is a $MZOP$ on PN_2 where $\alpha_1' = \lambda$, $\beta' = e$ and $(\alpha', \beta') = a$.

***Definition** 7:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$(i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. Let α, β be two paths on PN, then we define an operation "$\otimes$" representing the concurrent composition of paths on PN shown on the top of next page:

Remark 3: 1)$M \in [M_0 >$;
2) The definition is a recursive one since case 1) is applied to the latter cases;
3) We apply operation " $\otimes$ " to express concurrent composition of paths on PN;
4) We provide two rules for calculus of paths on PN as follows:
4.1)The operation degree of " $\otimes$ " is higher than " $\circ$ ";
4.2)The operation degree of " $\circ$ " is higher than " $+$ ".

***Definition** 8:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$(i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$.

1) Let $B^i_{M,M'}$=$\{\alpha \in (2^{T_i^*})^* |\ \alpha$ is a $BZOP_s$ on PN_i satisfying $M[\alpha > M'$ and $M \in [M_0 >\}$(i=1,2). $B^i_{M,M'}$ represents the set of all $BZOP_s$ on PN_i from M to M'.
2) Let $\overline{B}^i_{M,M'}$=$\{\alpha \in (2^{T_i^*})^* | \alpha$ is a $BNZOP$ on PN_i such that $M[\alpha > M'$ and $M \in [M_0 >\}$(i=1,2). $\overline{B}^i_{M,M'}$ represents the set of all $BNZOP_s$ on PN_i from M to M'.
3) Let $l_i(PN_i) = \{B^i_{M,M'} | \forall M \in [M_0 >, \forall M' \in [M >\}$, then $l_i(PN_i)$ represents the set of all $BZOP_s$ on PN_i (i=1,2).
4) Let $\overline{l_i}(PN_i) = \{\overline{B}^i_{M,M'} | \forall M \in [M_0 >, \forall M' \in [M >\}$, then $\overline{l_i}(PN_i)$ represents the set of all $BNZOP_s$ on PN_i (i=1,2).

***Definition** 9:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$(i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$.

1) If each $BNZOP$ on PN_i can be extended to a $BZOP$ on PN_i, then PN_i is called a well structured Petri net system($WSPNS$). i.e. for each $BNZOP$ α on PN_i, there exist $\alpha_1 \in (2^{(T_i \setminus T_0)^*})^*, R_t \in 2^{T_{0_i}^*}$ and $\beta \in (2^{(T_{3-i} \setminus T_{0_{3-i}})^*})^*$ such that $\alpha \circ \alpha_1 \circ R_t$ is a $BZOP$ on PN_i and β is its $MZOP$.
2) If PN_i is not a $WSPNS$, then PN_i is called a unwell structured Petri net system.

***Definition** 10:* Let $PN = (P, T; , F, M_0)$, be Petri net system and U,V the set of step sequences of PN. If the

$$\alpha\otimes\beta=\begin{cases}
\begin{pmatrix}\alpha_1\\ \beta\end{pmatrix}\circ(\alpha,\beta),\ \alpha_1\circ\beta\circ(\alpha,\beta). & \text{if } \alpha \text{ and } \beta \text{ are a pair of } MZOP \text{ on } PN, \text{ written}\\
 & \text{as } \alpha=\alpha_1\circ(\alpha,\beta), (\alpha_1 \text{ and } \beta \text{ can occur concurrently})\\
 & \text{if } \alpha \text{ and } \beta \text{ are not a pair of } MZOP \text{ on } PN,\\
 & \text{1) if } \alpha \text{ and } \beta \text{ are } BNZOP_s \text{ on } PN_i \text{ or } PN_{3-i},\\
\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \alpha\circ\beta. & \text{1.1) if } M[\alpha> M_1 \Rightarrow M_1[\beta> \text{ and } M[\beta> M_2 \Rightarrow\\
 & M_2[\alpha>, (\alpha \text{ and } \beta \text{ can occur concurrently})\\
\alpha\circ\beta. & \text{1.2) if } M[\alpha> M_1 \Rightarrow M_1[\beta> \text{ and } M[\beta> M_2 \Rightarrow\\
 & \neg M_2[\alpha>, (\alpha \text{ and } \beta \text{ occur in sequence})\\
\lambda. & \text{1.3) otherwise,}\\
 & \text{2) if } \alpha,\beta \text{ are } BNZOP \text{ and } BZOP \text{ on } PN_i \text{ or}\\
 & PN_{3-i} \text{ respectively, let } \gamma \text{ is an } MZOP \text{ of } \beta,\\
\begin{pmatrix}\alpha\\ \beta\otimes\gamma\end{pmatrix},\ \alpha\circ(\beta\otimes\gamma). & \text{2.1) if } M[\alpha> M_1 \Rightarrow M_1[\beta\otimes\gamma> \text{ and } M[\beta\otimes\gamma>\\
 & M_2 \Rightarrow M_2[\alpha>,\\
\alpha\circ(\beta\otimes\gamma). & \text{2.2) if } M[\alpha> M_1 \Rightarrow M_1[\beta\otimes\gamma> \text{ and } M[\beta\otimes\gamma>\\
 & M_2 \Rightarrow \neg M_2[\alpha>,\\
\lambda. & \text{2.3) if } M[\alpha> M_1 \Rightarrow \neg M_1[\beta\otimes\gamma>,\\
 & \text{3) if } \alpha,\ \beta \text{ are } BZOP \text{ and } BNZOP \text{ on } PN_i \text{ or}\\
 & PN_{3-i} \text{ respectively, let } \gamma \text{ is a } MZOP \text{ of } \alpha,\\
\begin{pmatrix}\alpha\otimes\gamma\\ \beta\end{pmatrix},\ (\alpha\otimes\gamma)\circ\beta. & \text{3.1) if } M[\alpha\otimes\gamma> M_1 \Rightarrow M_1[\beta> \text{ and } M[\beta> M_2\\
 & \Rightarrow M_2[\alpha\otimes\gamma>,\\
(\alpha\otimes\gamma)\circ\beta. & \text{3.2) if } M[\alpha\otimes\gamma> M_1 \Rightarrow M_1[\beta> \text{ and } M[\beta> M_2\\
 & \Rightarrow \neg M_2[\alpha\otimes\gamma>,\\
\lambda. & \text{3.3) otherwise,}\\
 & \text{4) if } \alpha \text{ and } \beta \text{ are both } BZOP_s \text{ on } PN_i \text{ or } PN_{3-i},\\
 & \text{let } \gamma,\ \sigma \text{ be } MZOP_s \text{ of } \alpha \text{ and } \beta \text{ respectively,}\\
\begin{pmatrix}\alpha\otimes\gamma\\ \beta\otimes\sigma\end{pmatrix},\ (\alpha\otimes\gamma)\circ(\beta\otimes\sigma). & \text{4.1) if } M[\alpha\otimes\gamma> M_1 \Rightarrow M_1[\beta\otimes\sigma> \text{ and } M[\beta\\
 & \otimes\sigma> M_2 \Rightarrow M_2[\alpha\otimes\gamma>,\\
(\alpha\otimes\gamma)\circ(\beta\otimes\sigma). & \text{4.2) if } M[\alpha\otimes\gamma> M_1 \Rightarrow M_1[\beta\otimes\sigma> \text{ and } M[\beta\\
 & \otimes\sigma> M_2 \Rightarrow \neg M_2[\alpha\otimes\gamma>,\\
\lambda. & \text{4.3) otherwise,}\\
 & \text{5) if } \alpha \not\subseteq (2^{T_i^*})^* \cup (2^{T_{3-i}^*})^* \text{ or } \beta \not\subseteq (2^{T_i^*})^* \cup\\
 & (2^{T_{3-i}^*})^*, (\text{both } \alpha \text{ and } \beta \text{ consist of sequences from } (2^{T_i^*})^* \text{ and } (2^{T_{3-i}^*})^*)\\
\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\ \alpha\circ\beta. & \text{5.1) if } M[\alpha> M_1 \Rightarrow M_1[\beta> \text{ and } M[\beta> M_2 \Rightarrow\\
 & M_2[\alpha>,\\
\alpha\circ\beta. & \text{5.2) if } M[\alpha> M_1 \Rightarrow M_1[\beta> \text{ and } M[\beta> M_2 \Rightarrow\\
 & \neg M_2[\alpha>,\\
\lambda. & \text{5.3) otherwise,}
\end{cases}$$

language $L(PN)$ of PN satisfies:

$$L(PN)=L(PN)\otimes U+V$$

then this equation is called a recursive language equation of PN or the language $L(PN)$ can be iteratively generated by this equation, where the initial value is $L^{(0)}(PN)=\{\lambda\}$.

Example 1: PN is synthesized with PN_1 and PN_2 by two inhibitor-arc connections, shown by figure 1(From left to right,it is PN_1, PN_2 and PN respectively).

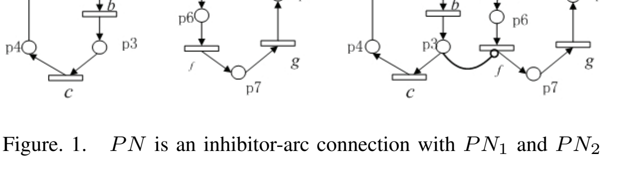

Figure. 1. PN is an inhibitor-arc connection with PN_1 and PN_2

It is easy to know that the set of all $BZOP_s$ on PN_1 is $l_1(PN_1)$=$\{a, da, cda,\ bcda\}$, and the set of all $BNZOP_s$ on PN_1 is $\bar{l}_1(PN_1)=\{\lambda, b, c, d, bc, cd, bcd\}$. Similarly, it is easy to get that the set of all $BZOP_s$ on PN_2 is $l_2(PN_2)=\{f, ef, hef, ghef\}$ and the set of all $BNZOP_s$ on PN_2 is $\bar{l}_2(PN_2)=\{\lambda, e, g, h, gh, he, ghe\}$

From figure 1, we have $\alpha = eaf\begin{pmatrix}g\\ bcd\end{pmatrix}a\begin{pmatrix}bc\\ he\end{pmatrix}f\begin{pmatrix}d\\ gh\end{pmatrix}$ is a sentence of $L_C(PN)$ and it can be shown that $\alpha=e\otimes a\otimes f\otimes\lambda\otimes g\otimes bcda\otimes bc\otimes hef\otimes d\otimes gh$, where $e\in\bar{l}_2(PN_2)$, $a\in l_1(PN_1)$, $f\in l_2(PN_2)$, $\lambda\in\bar{l}_1(PN_1)$, $g\in\bar{l}_2(PN_2)$, $bcda\in l_1(PN_1)$, $bc\in\bar{l}_1(PN_1)$, $hef\in l_2(PN_2)$, $d\in\bar{l}_1(PN_1)$ and $gh\in\bar{l}_2(PN_2)$. Because

$$\begin{aligned}
\alpha &= e\otimes a\otimes f\otimes\lambda\otimes g\otimes bcda\otimes bc\otimes hef\otimes d\otimes gh\\
&= e\circ a\otimes f\otimes\lambda\otimes g\otimes bcda\otimes bc\otimes hef\otimes d\otimes gh\\
&= e\circ a\circ f\otimes g\otimes bcda\otimes bc\otimes hef\otimes d\otimes gh\\
&= e\circ a\circ f\circ g\otimes bcda\otimes bc\otimes hef\otimes d\otimes gh\\
&= e\circ a\circ f\circ\begin{pmatrix}g\\ bcd\end{pmatrix}a\otimes bc\otimes hef\otimes d\otimes gh\\
&= e\circ a\circ f\circ\begin{pmatrix}g\\ bcd\end{pmatrix}a\circ bc\otimes hef\otimes d\otimes gh\\
&= e\circ a\circ f\circ\begin{pmatrix}g\\ bcd\end{pmatrix}a\circ\begin{pmatrix}bc\\ he\end{pmatrix}f\circ d\otimes gh\\
&= e\circ a\circ f\circ\begin{pmatrix}g\\ bcd\end{pmatrix}a\circ\begin{pmatrix}bc\\ he\end{pmatrix}f\circ\begin{pmatrix}d\\ gh\end{pmatrix}
\end{aligned}$$

$$= eaf\begin{pmatrix} g \\ bcd \end{pmatrix} a \begin{pmatrix} bc \\ he \end{pmatrix} f \begin{pmatrix} d \\ gh \end{pmatrix}$$

We can get that there are some pairs of $MZOP_s$ in α such as f,λ; g,bcda; bc,hef. Moreover, PN_1 and PN_2 are $WSPNS_s$, and $L_C(PN)$ is the set of sentences generated by the following recursive language equation, where the initially iterative value is $L^{(0)}(PN) = \lambda$.

$L_C(PN) = L_C(PN) \otimes (l_1(PN_1) \otimes \bar{l}_2(PN_2) + \bar{l}_2(PN_2) \otimes l_1(PN_1) + l_2(PN_2) \otimes \bar{l}_1(PN_1) + \bar{l}_1(PN_1) \otimes l_2(PN_2) + \bar{l}_1(PN_1) + \bar{l}_2(PN_2))$

III. A CONCURRENT LANGUAGE RELATION IN INHIBITOR-ARC CONNECTION OF PETRI NET SYSTEMS

This section will present a concurrent language based relation formula in inhibitor-arc connection of Petri net systems which represents the dynamic and concurrent behavior among these systems.

***Theorem** 1:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$(i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. Then the language $L_C(PN)$ of PN (For sake of brevity, we write $L(PN)$ as $L_C(PN)$ in the rest of this paper) satisfying the following recursive language equation:

$L(PN) = L(PN) \otimes (l_1(PN_1) \otimes \bar{l}_2(PN_2) + \bar{l}_2(PN_2) \otimes l_1(PN_1) + l_2(PN_2) \otimes \bar{l}_1(PN_1) + \bar{l}_1(PN_1) \otimes l_2(PN_2) + \bar{l}_1(PN_1) + \bar{l}_2(PN_2))$

where $l_i(PN_i)$, $\bar{l}_i(PN_i)$ (i=1,2) and $\otimes$ are defined by Definition 7, 8 respectively. We denote this recursive language equation as the recursive equation $(*)$.

Proof: We will prove the fact that for $\forall \alpha \in L(PN)$, α can be generated by the recursive equation $(*)$. Since PN is synthesized with PN_1 and PN_2 by the inhibitor-arc connections, we demonstrate it in following cases:

Case 1) $\alpha \cap 2^{T_0^*} = \emptyset$.

Case 1.1) α is a $BNZOP$ on PN_i (i=1,2), then α clearly can be generated by $L(PN) = L(PN) \otimes \bar{l}_i(PN_i)$. Thus, α can be produced from the recursive equation $(*)$.

Case 1.2) α is composed of $BNZOP_s$ on PN_i and PN_{3-i} alternately, then α can be similarly generated by $L(PN) = L(PN) \otimes (\bar{l}_i(PN_i) + \bar{l}_{3-i}(PN_{3-i}))$. Hence, α can be produced from the recursive equation $(*)$.

Case 2) $\alpha \cap 2^{T_0^*} \neq \emptyset$.

Case 2.1) α is composed of some pairs of $MZOP_s$, then α can be produced from $L(PN) = L(PN) \otimes (l_i(PN_i) \otimes \bar{l}_{3-i}(PN_{3-i}) + \bar{l}_{3-i}(PN_{3-i}) \otimes l_i(PN_i) + l_{3-i}(PN_{3-i}) \otimes \bar{l}_i(PN_i) + \bar{l}_i(PN_i) \otimes l_{3-i}(PN_{3-i}))$. Evidently, α can be generated by the recursive equation $(*)$.

Case 2.2) α is composed of some pairs of $MZOP_S$ and a $BNZOP$ on PN_i, then α can be generated by $L(PN) = L(PN) \otimes (l_i(PN_i) \otimes \bar{l}_{3-i}(PN_{3-i}) + \bar{l}_{3-i}(PN_{3-i}) \otimes l_i(PN_i) + l_{3-i}(PN_{3-i}) \otimes \bar{l}_i(PN_i) + \bar{l}_i(PN_i) \otimes l_{3-i}(PN_{3-i}) + \bar{l}_i(PN_i))$.

Case 2.3) α is composed of some pairs of $MZOP_s$ and $BNZOP_s$ on PN_i , PN_{3-i}, then α analogously can be produced from the recursive equation $(*)$. □

IV. THE PROPERTY OF DYNAMIC INVARIANCE IN INHIBITOR-ARC CONNECTION OF PETRI NET SYSTEMS

The concept of dynamic invariance including state and behavior invariance was first proposed in paper[4] on studying of synchronous and sharing synthesis processes. Their formal definitions are as follows:

***Definition** 11:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O PN_2 = (P, T; F, M_0)$, where O is a synthesis operation. If $\forall M \in [M_0 >, M|_{P_i} \in [M_i >$ (i=1,2), then the composite system PN satisfies state invariance.

***Definition** 12:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O PN_2 = (P, T; F, M_0)$, where O is a synthesis operation. If $\forall \alpha \in L(PN), \alpha|_{T_i} \in L(PN_i)$ (i=1,2), then the resultant system PN satisfies behavior invariance.

In paper[4], it showed that the dynamic invariance holds in a synchronous synthesis process except for sharing process. We now show that the synthesized system PN in inhibitor-arc connections with PN_1 and PN_2 also satisfies dynamic invariance.

***Theorem** 2:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$, then PN satisfies state invariance.

Proof: For $\forall M \in [M_0 >$, written as $M_0[\alpha > M$, We prove this theorem in following cases:

Case 1) if α is a $BNZOP$ on PN_i (i=1,2), obviously $M_0|_{P_i}[\alpha > M|_{P_i}$ and $M|_{P_{3-i}} = M_0|_{P_{3-i}}$. On the other hand, $M_0|_{P_i} = M_{0_i}$...(1) and $M_0|_{P_{3-i}} = M_{0_{3-i}}$...(2). Thus, $M|_{P_i} \in [M_{0_i} >$ and $M|_{P_{3-i}} \in [M_{0_{3-i}} >$.

Case 2) if α is composed of $BNZOP_s$ on PN_i and PN_{3-i} in turn, for sake of brevity, let $\alpha = \alpha_1 \otimes \alpha_2 (\neq \lambda)$, where α_1 and α_2 are $BNZOP_s$ on PN_i and PN_{3-i} respectively. We similarly obtain $M_0|_{P_i}[\alpha_1 > M|_{P_i}$ and $M_0|_{P_{3-i}}[\alpha_2 > M|_{P_{3-i}}$.With 1), 2) we have $M|_{P_i} \in [M_{0_i} >$ and $M|_{P_{3-i}} \in [M_{0_{3-i}} >$.

Case 3) if α is composed of some pairs of $MZOP_s$, let $\alpha = \alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_{2k-1} \otimes \alpha_{2k} (\neq \lambda)(k \geq 1)$, where α_{2j-1}, α_{2j} is a pair of $MZOP_s$ $(1 \leq j \leq k)$. By induction over k, we discuss α in following cases:

Case 3.1) if $k = 1, \alpha = \alpha_1 \otimes \alpha_2 (\neq \lambda)$. Suppose α_1 is a $BZOP$ on PN_i and α_2 is its $MZOP$, written as $\alpha_1 = \alpha_1^{'} \circ (\alpha_1, \alpha_2)$, then we get $M_0|_{P_i}[\alpha_1^{'} > M_1|_{P_i}$, $M_1|_{P_{3-i}}[\alpha_2 > M_2|_{P_{3-i}}$, $M_1|_{P_i} = M_2|_{P_i}$ and $M_2[(\alpha_1, \alpha_2) > M$. As PN is an inhibitor-arc connection net system of PN_i and PN_{3-i}, we have $M|_{P_{3-i}} = M_2|_{P_{3-i}}$, and $M_1|_{P_i}[(\alpha_1, \alpha_2) > M$ holds in single PN_i. Then

$M_0|_{P_i}[\alpha_1^{'} > M_2|_{P_i}[(\alpha_1, \alpha_2) >$ holds in single PN_i,

$M_0|_{P_{3-i}}[\alpha_2 > M|_{P_{3-i}}$ holds in single PN_{3-i}.

From 1), 2) we get $M|_{P_i} \in [M_{0_i} >$ and $M|_{P_{3-i}} \in [M_{0_{3-i}} >$.

Case 3.2) if $k = n$, the induction hypothesis is true.

Case 3.3) if $k = n + 1$, let $M_0[\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_{2n-1} \otimes \alpha_{2n} > M_1[\alpha_{2n+1} \otimes \alpha_{2n+2} > M$, where $\alpha_{2n+1} \otimes \alpha_{2n+2} \neq \lambda$. From the hypothesis, we have $M_1|_{P_i} \in [M_{0_i} >$ and $M_1|_{P_{3-i}} \in [M_{0_{3-i}} >$. Suppose that α_{2n+1} is a $BNZOP$ on PN_i and α_{2n+2} is its

$MZOP$, written as $\alpha_{2n+1} = \alpha'_{2n+1} \circ (\alpha_{2n+1}, \alpha_{2n+2})$, then we obtain $M_1|_{P_i}[\alpha'_{2n+1} > M_2|_{P_i}, M_2|_{P_{3-i}}[\alpha_{2n+2} > M_3|_{P_{3-i}}$, $M_1|_{P_{3-i}} = M_2|_{P_{3-i}}$, $M_3[(\alpha_{2n+1}, \alpha_{2n+2}) > M$ and $M_3|_{P_i} = M_2|_{P_i}$. Since PN is an inhibitor-arc connection net system of PN_i and PN_{3-i},, we have $M|_{P_{3-i}} = M_3|_{P_{3-i}}$, and $M_3|_{P_i}[(\alpha_{2n+1}, \alpha_{2n+2}) > M|_{P_i}$ holds in single PN_i. Hence, $M_1|_{P_i}[\alpha'_{2n+1} > M_3|_{P_i}[(\alpha_{2n+1}, \alpha_{2n+2}) > M|_{P_i}$ holds in single PN_i, and $M_1|_{P_{3-i}}[\alpha_{2n+2} > M|_{P_{3-i}}$ also holds in single PN_{3-i}. Note that $M_1|_{P_i} \in [M_{0_i} >$ and $M_1|_{P_{3-i}} \in [M_{0_{3-i}} >$, then we get $M|_{P_i} \in [M_{0_i} >$ and $M|_{P_{3-i}} \in [M_{0_{3-i}} >$.

Case 4) if α is composed of some pairs of $MZOP_s$ and a $BNZOP$ on PN_i, written as $\alpha = \alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_{2k-1} \otimes \alpha_{2k} \otimes \alpha_{2k+1} (\neq \lambda)$, where $\alpha_{2j-1}, \alpha_{2j}$ is a pair of $MZOP$ $(1 \leq j \leq k)$ and α_{2k+1} is a $BNZOP$ on PN_i. Let $M_0[\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_{2k-1} \otimes \alpha_{2k} > M'[\alpha_{2k+1} > M$. From case 3), we have $M'|_{P_i} \in [M_{0_i} >$ and $M'|_{P_{3-i}} \in [M_{0_{3-i}} >$. As α_{2k+1} is a $BNZOP$ on PN_i, $M'|_{P_i}[\alpha_{2k+1} > M|_{P_i}$ and $M|_{P_{3-i}} = M'|_{P_{3-i}}$. Then we obtain $M|_{P_i} \in [M_{0_i} >$ and $M|_{P_{3-i}} \in [M_{0_{3-i}} >$.

Case 5) if α is composed of some pairs of $MZOP_s$ and $BNZOP_s$ on PN_i , PN_{3-i}, with similarity to case 4), the conclusion can be easily proved. □

The synthesized system $PN = PN_1 O_I PN_2$ also satisfies behavior invariance.

***Theorem** 3:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$, then PN satisfies behavior invariance.

Proof: This theorem in fact has been proved in the proof of theorem 2. □

V. Liveness in inhibitor-arc connection operations

In this paragraph, we apply the above results to judge the liveness of synthesized system in inhibitor-arc connection operation.

***Definition** 13:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. The concurrent language generated by the following recursive equation:

$L(PN) = L(PN) \otimes (l_1(PN_1) \otimes \bar{l}_2(PN_2) + \bar{l}_2(PN_2) \otimes l_1(PN_1) + l_2(PN_2) \otimes \bar{l}_1(PN_1) + \bar{l}_1(PN_1) \otimes l_2(PN_2))$ is called the kernel of $L(PN)$, where the initial value is $L^{(0)}(PN) = \lambda$, denoted as $Ker(L(PN))$.

***Definition** 14:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = \Sigma_1 O_I PN_2 = (P, T; F, M_0)$ and $L \subseteq L(PN)$. Let

$$\overline{L} = \{\alpha \in L | \exists \alpha' \in L, \alpha \otimes \alpha' (\neq \lambda) \in L\};$$
$$\overrightarrow{L} = \{\alpha \in L | \exists \alpha' \in L, \alpha \otimes \alpha' (\neq \lambda) \in L \; and \; |\alpha'| \neq 0\};$$
$$\overline{\overline{L}} = \{\alpha \in L | \exists \alpha' \in L, \alpha \otimes \alpha' (\neq \lambda) \in L \; and \; \|\alpha'\| = \|L\|\}.$$

where $|\alpha|$ represents the number of characters occurring in α, $\|\alpha\|$ indicates all the characters occurring in α. Then $\overline{L}$, $\overrightarrow{L}$ and $\overline{\overline{L}}$ are called the recursive closure, strict recursive closure and strong recursive closure of L respectively.

***Theorem** 4:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. If PN_1 or PN_2 is not a $WSPNS$, then PN is not live.

Proof: Without loss of generality, suppose PN_1 is not a $WSPNS$, then there exists a $BNZOP$ α on PN_1 which can not be extended to a $BZOP$ on PN_1. Let $M[\alpha > M'$, where $M \in [M_0 >$, then for $\forall R_t \in 2^{T^*_{0_1}}$ and $\forall M'' \in [M' >$, we have $\neg M''[R_t >$. Hence, PN is not live. □

***Theorem** 5:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. PN is live iff

1) PN_1 and PN_2 are both $WSPNS_s$;
2) $Ker(L(PN)) = \overline{\overline{Ker(L(PN))}}$.

Proof: “ $\Rightarrow$ ” If PN is live, from Theorem 4, condition 1) should hold. For $\forall \alpha \in Ker(L(PN)) (\alpha \neq \lambda)$ and $\forall t \in T$, let $M[\alpha > M'$, where $M \in [M_0 >$. We show the fact that there exists $\alpha' \in Ker(L(PN))$ such that $\alpha \otimes \alpha' (\neq \lambda) \in Ker(L(PN))$ and $t \in \alpha'$. From the definition of $Ker(L(PN))$, let $\alpha = \alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_{2k-1} \otimes \alpha_{2k} (\neq \lambda)$, where $\alpha_{2j-1}, \alpha_{2j}$ are a pair of $MZOP_s$ $(1 \leq j \leq k)$. As PN_1 and PN_2 are both $WSPNS_s$, t must lie in a pair of $MZOP_s$. Let $\alpha \otimes \alpha'_1 \otimes \alpha'_2 \otimes ... \otimes \alpha'_{2n-1} \otimes \alpha'_{2n} (\neq \lambda)$ be an arbitrary chain of $MZOP_s$ starting from α. If α' does not lie in this chain, we have $\forall M'' \in [M' >$: $\neg M''[t >$. A contradiction with the liveness of PN. Thus there exists $\alpha' \in Ker(L(PN))$ such that $\alpha \otimes \alpha' (\neq \lambda) \in \overline{\overline{Ker(L(PN))}}$ and $\|\alpha'\| = T$. Hence, $Ker(L(PN)) \subseteq \overline{\overline{Ker(L(PN))}}$. On the other hand, $\overline{\overline{Ker(L(PN))}} \subseteq Ker(L(\Sigma))$. Therefore, $Ker(L(PN)) = \overline{\overline{Ker(L(PN))}}$, then condition 2) hold.

“ $\Leftarrow$ ” If 1) and 2) hold, for $\forall M \in [M_0 >$, $\forall t \in T$, let $M_0[\alpha > M$, we prove that $\exists M' \in [M >$ such that $M'[t >$.

1) If α is a $BNZOP$ on PN_i, since PN_i is a $WSPNS$, there exist $\alpha_1 \in (2^{T^*_i})^*$ and $\alpha_2 \in (2^{(T_{3-i} \setminus T_{0_{3-i}})^*})^*$ such that $\alpha \otimes \alpha_1 (\neq \lambda)$ and α_2 are a pair of $MZOP$. Then we obtain $\alpha \otimes \alpha_1 \otimes \alpha_2 (\neq \lambda) \in Ker(L(PN))$. On the other hand, $Ker(L(PN)) = \overline{\overline{Ker(L(PN))}}$, there exists $\alpha_3 \in Ker(PN))$ with $\|\alpha_3\| = T$, written as $\alpha_3 = \alpha'_3 \circ t \circ \alpha''_3$, and $\alpha \otimes \alpha_1 \otimes \alpha_2 \otimes \alpha_3 (\neq \lambda) \in Ker(L(PN))$. Then $\exists M_2 \in [M >$ satisfying $M[\alpha \otimes \alpha_1 \otimes \alpha_2 > M_1[\alpha'_3 > M_2[t >$.

2) If α is composed of some pairs of $BNZOP_s$ on PN_i and PN_{3-i} alternately, the conclusion can be proved similarly.

3) If α is composed of some pairs of $MZOP_s$, it is easy to obtain that the conclusion is true.

4) If α is composed of some pairs of $MZOP_s$ and $BNZOP_s$ on PN_i, PN_{3-i}, the conclusion can be verified analogously.

Summarizing the above discussion, we get that PN is live. □

VI. Liveness preservation in inhibitor-arc connection operations

We present in this section some conditions under which each subsystem is live iff the synthesized system is live.

***Theorem** 6:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. PN is live iff

1) PN_i (i=1,2) are both $WSPNS_s$;
2) PN_i (i=1,2) are both live.

Proof: " $\Leftarrow$ " We apply Theorem 5 to prove this conclusion. From condition 1), we need to prove $Ker(L(PN)) = \overline{\overline{Ker(L(PN))}}$. Due to $\overline{\overline{Ker(L(PN))}} \subseteq Ker(L(PN))$, we only need to show $Ker(L(PN)) \subseteq \overline{\overline{Ker(L(PN))}}$. For $\forall \alpha \in Ker(L(PN))$, let $\alpha = \alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_{2k-1} \otimes \alpha_{2k} (\neq \lambda)$, where $\alpha_{2j-1}, \alpha_{2j}$ are a pair of $MZOP_s$ $(1 \leq j \leq k)$. Without loss of generality, suppose α_{2k-1} is a $BZOP$ on PN_1 and α_{2k} is its $MZOP$ on PN_2, written as $M_1[\alpha_{2k-1} \otimes \alpha_{2k} > M_2$, where $M_1 \in [M_0 >$. By Theorem 2, we have $M_2|_{P_2} \in [M_{0_2} >$. As PN_2 is live and is a $WSPNS$, for $\forall t \in T_2$, there exists $\alpha^{'} \in (2^{T_2^*})^*$, written as $\alpha^{'} = \alpha_1^{'} \circ \alpha_2^{'} \circ ... \circ \alpha_n^{'}$, where $\alpha_j^{'} (1 \leq j \leq n-1)$ are $BZOP_s$ on PN_2 and $\alpha_n^{'}$ is a $BNZOP$ or $BZOP$ on PN_2 (suppose $\alpha_n^{'}$ is a $BNZOP$ on PN_2), such that $M_2|_{P_2}[\alpha_1^{'} \circ \alpha_2^{'} \circ ... \circ \alpha_n^{'} \circ t >$ holds on single PN_2. Meanwhile, there exists $\alpha_i^{''}$, which is a $MZOP$ of $\alpha_i^{'}$ on PN_1 satisfying $\alpha_i^{'} \otimes \alpha_i^{''} \neq \lambda$. Then $\exists M_3 \in [M_2 >$ with $M_2[\alpha_1^{'} \otimes \alpha_1^{''} \otimes ... \otimes \alpha_{n-1}^{'} \otimes \alpha_{n-1}^{''} > M_3$ and $M_3[\alpha_n^{'} >$.

If $\alpha_n^{'} \circ t$ is a $BZOP$ on PN_2, i.e. $t \in T_{0_2}$, let $\alpha_n^{''}$ be its $MZOP$ on PN_1, we obtain $\overline{\alpha} = \alpha_1^{'} \otimes \alpha_1^{''} \otimes ... \otimes \alpha_{n-1}^{'} \otimes \alpha_{n-1}^{''} \otimes (\alpha_n^{'} \circ t) \otimes \alpha_n^{''} \in Ker(L(PN))$ satisfying $\alpha \otimes \overline{\alpha} (\neq \lambda) \in Ker(L(PN))$ and $t \in \overline{\alpha}$. If $\alpha_n^{'} \circ t$ is a $BNZOP$ on PN_2, as PN_2 is a $WSPNS$, there exist $\alpha_{n+1}^{'} \in (2^{T_2^*})^*$ and $\alpha_{n+1}^{''} \in (2^{(T_1 \backslash T_{0_1})^*})^*$ such that $\alpha_n^{'} \circ t \circ \alpha_{n+1}^{'}$ is a $BZOP$ on PN_2 and $\alpha_{n+1}^{''}$ is its $MZOP$ on PN_1. Then we get $\overline{\alpha} = \alpha_1^{'} \otimes \alpha_1^{''} \otimes ... \otimes \alpha_{n-1}^{'} \otimes \alpha_{n-1}^{''} \otimes (\alpha_n^{'} \circ t \circ \alpha_{n+1}^{'}) \otimes \alpha_{n+1}^{''} (\neq \lambda) \in Ker(L(PN))$ satisfying $\alpha \otimes \overline{\alpha} (\neq \lambda) \in Ker(L(PN))$ and $t \in \overline{\alpha}$. In brief, there must exist $\overline{\alpha} \in Ker(L(PN))$ with $\|\overline{\alpha}\| = T_2$ satisfying $\alpha \otimes \overline{\alpha} (\neq \lambda) \in Ker(L(PN))$.

Similarly, there exists $\overline{\alpha}^{'} \in Ker(L(PN))$ with $\|\overline{\alpha}^{'}\| = T_1$, verifying $\alpha \otimes \overline{\alpha} \otimes \overline{\alpha}^{'} (\neq \lambda) \in Ker(L(PN))$. Thus, we obtain $\alpha_0 = \overline{\alpha} \otimes \overline{\alpha}^{'}$ with $\|\alpha_0\| = T$ such that $\alpha \otimes \alpha_0 (\neq \lambda) \in Ker(L(PN))$. Hence, $\alpha \in \overline{\overline{Ker(L(PN))}}$ and $Ker(L(PN)) \subseteq \overline{\overline{Ker(L(PN))}}$. By Theorem 5, PN is live.

" $\Rightarrow$ " Since PN is live, from Theorem 5, PN_i(i=1,2) are $WSPNS_s$, we now show that the inhibitor-arc connection operation has no effect on sequential occurrence of every transition of PN_i (i=1,2) except for concurrent occurrence. For an arbitrary $BNZOP$ α_1 on PN_i, written as $M_1|_{P_i}[\alpha_1 > M_2|_{P_i}$ $(M_1 \in [M_0 >)$, satisfying $\exists t^i \in T_{0_i}$ such that $M_2|_{P_2}[t^i >$ holds on single PN_i. We first have $M_2|_{P_i} \in [M_{0_i} >$ from Theorem 2. Due to PN_i being a $WSPNS$, there exist $\alpha_2 \in (2^{T_{3-i}^*})^*$ and $M_3 \in [M_2 >$ such that $M_2|_{P_{3-i}}[\alpha_2 > M_3|_{P_{3-i}}$ and $M_3[t^i >$. Note that $M_3|_{P_i} \in [M_{0_i} >$ from Theorem 2, we have that $M_2|_{P_i}[t^i > M_3|_{P_i}$ holds on single PN_i.

Summarizing the above discussion, if $t^i \in T_{0_i}$ can occur on PN, it can occur in PN_i (If $t^i \in T_i \backslash T_{0_i}$, this conclusion is trivial).

Since PN is live, PN_i(i=1,2) are live. □

Remark 4: Theorem 5 and 6 present the necessary and sufficient conditions for judging liveness of synthesized system through liveness of its subsystems in inhibitor-arc connections. However, these conditions cannot be easily determined whether or not they hold in practice. We will present some sufficient conditions to judge the liveness of global system when subsystems are live.

***Definition** 15:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. For an arbitrary inhibitor arc $(p^i, t^{3-i}) \in F$, where $p^i \in P_i, t^{3-i} \in T_{3-i}$, then p^i is called an inhibiting place of t^{3-i}.

***Definition** 16:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. For an arbitrary inhibitor arc $(p^i, t^{3-i}) \in F$, where $p^i \in P_i, t^{3-i} \in T_{3-i}$, if $|^\bullet(t^{3-i}) \cap P_i| = 1$ (the dot denotation refers to PN), then PN satisfies one inhibiting place constraint connection condition.

***Definition** 17:* (No Self-loop Condition) Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. For an arbitrary inhibiting place $p \in P_i$(i=1,2), there is no self-loop between p and t, where $t \in p^\bullet$ (the dot denotation refers to PN_i). Then PN satisfies no self-loop condition.

Example 2. PN is an inhibitor-arc connection net system of PN_1 and PN_2, shown by figure 2 (From left to right, it is PN_1, PN_2 and PN respectively).

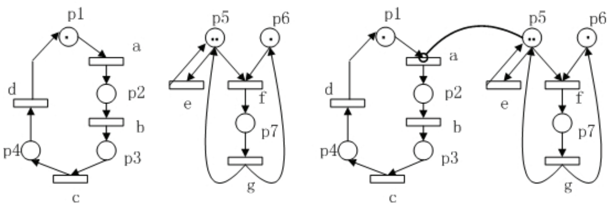

Figure. 2. PN does not satisfy no self-loop condition

It is easily known that inhibiting place p_5 does not satisfy no self-loop condition, and PN is not live even though PN_1, PN_2 are live.

***Lemma** 1:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. For $p \in P_i$ (i=1,2), if p satisfies the following three conditions, then all tokens can be removed from p through a path in which there is no transition belongs to T_{0_i} (i.e. $\forall M \in [M_0 >, \exists M^{'} \in [M >, \alpha \in (2^{(T_i \backslash T_{0_i})^*})^*$ such that $M[\alpha > M^{'}$ and $M^{'}(p) = 0$).

1) $p^\bullet \neq \emptyset$ and if $|p^\bullet| > 0$ then $|^\bullet(p^\bullet)| = 1$ (the dot denotation refers to PN_i).
2) there is no self-loop between p and t for $\forall t \in p^\bullet$ (the dot denotation refers to PN_i).

3) $\exists t \in p^\bullet$, t has no inhibiting place (the dot denotation refers to PN_i).

Proof: It is easily proved. □

***Theorem** 7:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. If PN_i (i=1,2) satisfy the following conditions, then $\{M|_{P_i} | \forall M \in [M_0 >\} = [M_{0_i} >$ (This property is called the strong state invariance).

1) PN satisfies one inhibiting place constraint connection condition.
2) All inhibiting places of PN satisfy no self-loop condition.
3) For every inhibiting place $p \in P_i$ (i=1,2), $p^\bullet \neq \emptyset$ and if $|p^\bullet| > 0$ then $|^\bullet(p^\bullet)| = 1$ (the dot denotation refers to PN_i).
4) For every inhibiting place $p \in P_i$ (i=1,2), $\exists t \in p^\bullet$ such that $t \notin T_{0_i}$ (the dot denotation refers to PN_i).

Proof: Obviously, from Theorem 2, $\{M|_{P_i} | \forall M \in [M_0 >\} \subseteq [M_{0_i} >$. Suppose $\exists M^i \in [M_{0_i} >$ (i=1,2) such that $M^i \notin \{M|_{P_i} | \forall M \in [M_0 >\}$. Let $M_{0_i}[\alpha > M^i$ on single PN_i, written as $\alpha = t_1 t_2 ... t_n (n \geq 1)$, where $t_j \in T_i (1 \leq j \leq n)$. For sake of brevity, let t_h is the unique transition which belongs to T_{0_i} and $M_{0_i}[t_1...t_{h-1} > M_1^i [t_h > M_2^i [t_{h+1}...t_n > M^i$. Let p^h is its unique inhibiting place in PN_{3-i}.

Note that $M_0|P_i = M_{0_i}$, we first fire transitions of T_i. Let $M_0|_{P_i}[t_1...t_{h-1} > M_1|_{P_i}$. Obviously, $M_1|_{P_i} = M_1^i$ and $\exists \alpha' \in (T_{3-i} \backslash T_{0_{3-i}})^*$ such that $M_1|_{P_{3-i}}[\alpha' > M_2|_{P_{3-i}}$ and $M_2(p^h) = 0$ from Lemma 1. Thus $M_2[t_h >$ and $M_2|_{P_i} = M_1|_{P_i}$. Let $M_2[t_h > M_3$, clearly, $M_3|_{P_i} = M_2^i$, and let $M_3|_{P_i}[t_{h+1}...t_n > M_4|_{P_i}$. Due to $M_3|_{P_i} = M_2^i$, we have $M^i = M_4|_{P_i}$, where $M_4 \in [M_0 >$. A contradiction with above supposition. Thus, $[M_{0_i} > \subseteq \{M|_{P_i} | \forall M \in [M_0 >\}$. Hence, $\{M|_{P_i} | \forall M \in [M_0 >\} = [M_{0_i} >$.

If α consists of more than one transitions which belongs to T_{0_i}, the conclusion can be similarly proved. □

***Theorem** 8:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. If the following conditions hold, then PN is live iff PN_1 and PN_2 are live.

1) PN satisfies one inhibiting place constraint connection condition.
2) All inhibiting places of PN satisfy no self-loop condition.
3) For every inhibiting place $p \in P_i$ (i=1,2), $p^\bullet \neq \emptyset$ and if $|p^\bullet| > 0$ then $|^\bullet(p^\bullet)| = 1$ (the dot denotation refers to PN_i).
4) For every inhibiting place $p \in P_i$ (i=1,2), $\exists t \in p^\bullet$ such that $t \notin T_{0_i}$ (the dot denotation refers to PN_i).

Proof: By Theorem 6, we need to prove PN_1, PN_2 are $WSPNS_s$, we first show PN_1 is a $WSPNS$. For an arbitrary $BNZOP$ α on PN_1, written as $M[\alpha > M_1$, due to the liveness of PN_1, $\exists \alpha_1 \in (2^{(T_1 \backslash T_{0_1})^*})^*$ (α_1 may might be λ), $M_2 \in [M_1 >$ and $t^1 \in T_{0_1}$ such that $M_1|_{P_1}[\alpha_1 > M_2|_{P_1}[t^1 >$ holds on single PN_1. From condition 2), there exists unique inhibiting place p^2 of t^1, where $p^2 \in P_2$. If $M_2(p^2) = 0$, then $M_2[t^1 >$ holds on PN and $\alpha \otimes \alpha_1 \circ t^1$ is a $BZOP$ on PN_1, λ is its $MZOP$. If not, due to conditions 3), 4) and Lemma 1, $\exists M_3 \in [M_2 >$ and $\alpha_2 \in (2^{(T_2 \backslash T_{0_2})^*})^*$ such that $M_2|_{P_2}[\alpha_2 > M_3|_{P_2}$ and $M_3(p^2) = 0$. Thus, $\alpha \otimes \alpha_1 \circ t^1$ is a $BZOP$ on PN_1 and α_2 is its $MZOP$. Hence, PN_1 is $WSPNS$.

Symmetrically, PN_2 can be proved to be a $WSPNS$. Therefore, PN is live iff PN_1 and PN_2 are live by Theorem 6. □

Example 3. PN is an inhibitor-arc connection net system of PN_1 and PN_2, shown by figure 3.

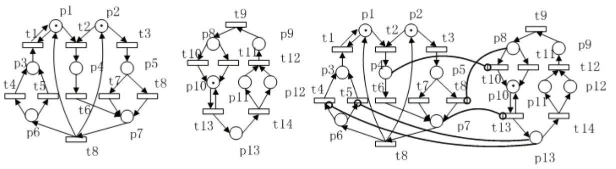

Figure. 3. PN satisfies all conditions of Theorem 8

In figure 3, it can be verified that PN satisfies conditions $1) - 4)$ of Theorem 8 and PN is live iff PN_1, PN_2 are live.

***Corollary** 1:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be strong connected state machines, where $|P_i| > 1$. $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. If the following conditions hold, then PN is live iff PN_1 and PN_2 are live.

1) PN satisfies one inhibiting place constraint connection condition.
2) For every inhibiting place $p \in P_i$ (i=1,2), $\exists t \in p^\bullet$ such that $t \notin T_{0_i}$ (the dot denotation refers to PN_i).

Proof: Since PN_i (i=1,2) are strong connected state machines and $|P_i| > 1$, conditions $1) - 2)$ imply conditions $1) - 4)$ of Theorem 8. Hence, PN is live iff PN_1 and PN_2 are live. □

***Theorem** 9:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be Petri net systems, $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. If the following conditions hold, then PN is live iff PN_1 and PN_2 are live.

1) $T_{0_i} = \emptyset$ and $T_{0_{3-i}} \neq \emptyset$ (i=1,2).
2) PN satisfies one inhibiting place constraint connection condition.
3) All inhibiting places of PN satisfy no self-loop condition.
4) For every inhibiting place $p \in P_i$ (i=1,2), $p^\bullet \neq \emptyset$ and if $|p^\bullet| > 0$ then $|^\bullet(p^\bullet)| = 1$ (the dot denotation refers to PN_i).

Proof: Conditions $1) - 4)$ imply conditions $1) - 4)$ of Theorem 8. Hence, PN is live iff PN_1 and PN_2 are live. □

Remark 5: Condition 1) means that PN_{3-i} is controlled by PN_i while PN_i is free.

***Corollary** 2:* Let $PN_i = (P_i, T_i; F_i, M_{0_i})$ (i=1,2) be strong connected state machines, where $|P_i| > 1$. $PN = PN_1 O_I PN_2 = (P, T; F, M_0)$. If the following conditions hold, then PN is live iff PN_1 and PN_2 are live.

1) $T_{0_i} = \emptyset$ and $T_{0_{3-i}} \neq \emptyset$ (i=1,2).

2) PN satisfies one inhibiting place constraint connection condition.

Proof: It is easily proved by Theorem 9. □

VII. Conclusion

The main analysis technique for critical systems is state space exploration. However, this is often limited by the so-called state space explosion problem. Many approaches have been devoted to tackle this problem in order to get suitable state space. Prominent among these are compositional methods, which take advantages of the modular structure of the model to build manageable state space indicating the global behavior. The contribution of this paper is a proposal of some criteria which are necessary and sufficient for the preservation of liveness in inhibitor-arc connection, i.e., conditions under which the liveness of synthesized system identifies the liveness of local systems. Contrasting with other works, our work is based on concurrent language. We present an operation " $\otimes$ " to express the concurrent composition of paths and establish a recursive language equation to judge the liveness of composite system. We also demonstrate an important property, namely, dynamic invariance in inhibitor-arc connection of net systems. One of the main advantages of our approaches is that we can synthesize Petri net systems beyond asymmetric choice net systems, thus it must have more effective applications. In particular, the approaches presented here can easily be generalized to Petri net systems with weighted arcs. Further directions we intend to investigate are the study of giving an extension to conditions of liveness preservation and study other properties preservation in inhibitor-arc connection or in another synthesis operation such as synchronous operation.

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