

# Item Relational Structure Algorithm Based on Empirical Distribution Critical Value

Hsiang-Chuan Liu \*

Department of Bioinformatics and Medical Informatics, Asia University, Taichung, Taiwan  
 Graduate Institute of Acupuncture Science, China Medical University, Taichung, Taiwan  
 Email: lhc@asia.edu.tw

Shih-Neng Wu

Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan  
 Email: wsneng@mail2000.com.tw

Chin-Chun Chen

Department of General Education, Min-Hwei College, Tainan, Taiwan  
 Email: ccchen@mail.mhchcm.edu.tw

**Abstract**—Takeya's item relational structure theory is a well-known ordering algorithm. However, its threshold limit value is a fixed value, lacking of statistical meaning. In this paper, the authors provide an improved threshold limit value by using the empirical distribution critical value of all the values of the relational structure indices between any two items, it is more sensitive and effective than the traditional fixed threshold for comparing the ordering relation of any two items. A computer program is developed for the proposed method. A Mathematics example is also provided in this paper to illustrate the advantages of the proposed methods.

**Index Terms**—item ordering theory, item relational structure theory, empirical distribution critical value

## I. INTRODUCTION

All of the correlation, distance and similarity, are symmetrical relations, which can not be used to detect item ordering relationships or directed structures of a group of subjects. There are two well-known ordering item algorithms based on the testing performance of a group of subjects, one is Ordering theory (OT) proposed by Bart et al in 1973 [1-2], and the other is Item relational structure theory (IRS) proposed by Takeya in 1991 [3-4], the former is not like the later considering not only the partial ordering relation but also the correlation relation. However, the threshold limit values of both of them are fixed values, lacking of statistical meaning, one of the authors of this paper, H.-C. Liu, transferred the ordering index as a approximated t value to obtained a critical t value at significant level 0.05 [5], but it is only used for

nominal scale data, not for interval scale data. In this paper, the authors consider to improve the threshold limit value by using the empirical distribution critical value of all the values of the relational structure indices between any two items, it is more valid than the traditional threshold for comparing the ordering relation of any two items. A computer program is developed for the proposed method. A Mathematics example is also provided in this paper to illustrate the advantages of the proposed methods.

## II. ORDERING THEORY ALGORITHM

In this section, Ordering Theory algorithm is described briefly as follows;

### A. Definition of Ordering Theory Algorithm

Airasian & Bart, 1973; Bart & Krus, 1973) [1-2] provided the Ordering Theory (OT) algorithm as following definition;

**Definition 1:** Ordering Theory (OT) algorithm

Let  $\varepsilon \in [0.02, 0.04]$ , and  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  denote a vector containing n binary item scores variables. Each individual taking n-item test produces a vector  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  containing ones and zeros. Then the joint and marginal probabilities of items j and k can be represented in table 1.

- (i) if  $P(X_j = 0, X_k = 1) < \varepsilon$ , then we say that item j could be linked forwards to item k. The relation is denoted as  $X_j \rightarrow X_k$ , it means that  $X_j$  is a prerequisite to  $X_k$
- (ii) if  $P(X_j = 0, X_k = 1) \geq \varepsilon$ , then we say that item j could not be linked forwards to item k.

\*corresponding author;

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The relation is denoted as  $X_j \not\Rightarrow X_k$ , it means that  $X_j$  is not a prerequisite to  $X_k$

- (iii) If  $X_j \rightarrow X_k$  and  $X_k \rightarrow X_j$ , then the relation is denoted as  $X_j \leftrightarrow X_k$  and it means that item j and k are equivalent. For convenience,
- (iv) If  $X_j \not\rightarrow X_k$  or  $X_k \not\rightarrow X_j$  then the relation is denoted as  $X_j \nleftrightarrow X_k$  and it means that item j and k are not equivalent.

For convenience, let  $\epsilon = 0.03$  in this paper.

Table 1 The joint probabilities of item j and k

		Item k		Total
		$X_k = 1$	$X_k = 0$	
Item j	$X_j = 1$	$P(X_j=1, X_k=1)$	$P(X_j=1, X_k=0)$	$P(X_j=1)$
	$X_j = 0$	$P(X_j=0, X_k=1)$	$P(X_j=0, X_k=0)$	$P(X_j=0)$
	Total	$P(X_k=1)$	$P(X_k=0)$	1

**B. Examples of Ordering Theory Algorithm**

**Example 1:**

Let the joint and marginal probabilities of item j and k of 122 subjects are listed as Table 2

Table 2: The joint probabilities of item i and j

		Item j		Total
		$X_j = 1$	$X_j = 0$	
Item i	$X_i = 1$	0.02	0.02	0.04
	$X_i = 0$	0.48	0.48	0.96
	Total	0.5	0.5	1

In Table 2, from the Definition 1, we can obtain following two results;

$$P(X_i=0, X_j=1) = 0.48 \geq \epsilon = 0.03 \Rightarrow X_i \not\rightarrow X_j \quad (1)$$

$$P(X_j=0, X_i=1) = 0.02 < \epsilon = 0.03 \Rightarrow X_j \rightarrow X_i, \quad (2)$$

Result (1) means that  $X_i$  and  $X_j$  have no ordering relation from  $X_i$  to  $X_j$ , but result (2) means that  $X_j$  and  $X_i$  have ordering relation from  $X_j$  to  $X_i$ . This fact

shows that item ordering relation is not a symmetric relation rather than correlation relation.

In addition, we can obtain the correlation value of  $X_j$  and  $X_i$  by using the formula (3), its value is equal to zero, that is  $\rho_{ij} = \rho_{ji} = 0$ , it means that  $X_j$  and  $X_i$  have no linear relationship, but from equation (2),  $X_j$  and  $X_i$  have ordering relation, it leads to a contradiction, in other words, Ordering Theory algorithm is not a valid ordering algorithm.

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \quad (3)$$

Where

$$\sigma_{xy} = \sum_{x,y} (x - \mu_x)(y - \mu_y) f_{XY}(x, y), \quad (4)$$

$$\sigma_x^2 = \sum_x (x - \mu_x)^2 f_X(x), \quad (5)$$

$$\sigma_y^2 = \sum_y (y - \mu_y)^2 f_Y(y) \quad (6)$$

**Example 2:**

Let the joint and marginal probabilities of item j and k of 122 subjects are listed as Table 3.

Table 3: The joint probabilities of item j and k

		Item k		Total
		$X_k = 1$	$X_k = 0$	
Item j	$X_j = 1$	0.50	0.05	0.55
	$X_j = 0$	0.05	0.40	0.45
	Total	0.55	0.45	1

In Table 3,

$$P(X_j=0, X_k=1) = 0.05 \geq \epsilon = 0.03 \Rightarrow X_j \not\Rightarrow X_k \quad (7)$$

$$P(X_k=0, X_j=1) = 0.05 > \epsilon = 0.03 \Rightarrow X_k \not\Rightarrow X_j \quad (8)$$

, but the correlation value of  $X_j, X_k$  is  $\rho_{jk} = 0.798$ , it means that  $X_j$  and  $X_k$  have high correlated linear relationship but no ordering relation, it also leads to a contradiction, in other words, Ordering Theory algorithm is indeed not a valid ordering algorithm.

**III. ITEM RELATIONAL STRUCTURE THEORY ALGORITHM**

For improving above-mentioned drawback, we need to consider both of ordering relation and linear correlation, since if and only if

$$P(X_j=0, X_k=1) = P(X_j=0)P(X_k=1), \quad (9)$$

then  $[X_j = 0]^{++} [X_k = 1]$  (10)

if  $P(X_j = 0)P(X_k = 1) > 0$  (11)

then

$$1 - \frac{P(X_j = 0, X_k = 1)}{P(X_j = 0)P(X_k = 1)} = 0 \Leftrightarrow [X_j = 0]^{++} [X_k = 1] \quad (12)$$

in other words,  $1 - \frac{P(X_j = 0, X_k = 1)}{P(X_j = 0)P(X_k = 1)}$  can be view as

the degree of correlation between events  $[X_j = 0]$  and  $[X_k = 1]$ , hence, Takeya (1991) proposed his improving item ordering theory algorithm called item relational structure as following definition;

A. Definition of Item Relational Structure Algorithm

**Definition 2:** Item Relational Structure (IRS) algorithm  
Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  denote a vector containing n binary item scores variables. Each individual taking n-item test produces a vector  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  containing ones and zeros. Then the joint and marginal probabilities of items j and k can be represented in Table 1

and 
$$r_{jk}^* = 1 - \frac{P(X_j = 0, X_k = 1)}{P(X_j = 0)P(X_k = 1)} \quad (13)$$

- (i) if  $r_{jk}^* > 0.5$ , then we say that item j could be linked forwards to item k. The relation is denoted as  $X_j \rightarrow X_k$ , it means that  $X_j$  is a prerequisite to  $X_k$
- (ii) if  $r_{jk}^* \leq 0.5$ , then we say that item j could not be linked forwards to item k. The relation is denoted as  $X_j \not\rightarrow X_k$ , it means that  $X_j$  is not a prerequisite to  $X_k$
- (iii) If  $X_j \rightarrow X_k$  and  $X_k \rightarrow X_j$ , then the relation is denoted as  $X_j \leftrightarrow X_k$  and it means that item j and k are equivalent. For convenience,
- (iv) If  $X_j \not\rightarrow X_k$  or  $X_k \not\rightarrow X_j$  then the relation is denoted as  $X_j \nleftrightarrow X_k$  and it means that item j and k are not equivalent.

B. Examples of Item Relational Structure Algorithm

**Example 3:** the data is the same as Example 1. In Table 2, from Example 1. we know that  $\rho_{ij} = \rho_{ji} = 0$ ,

And 
$$r_{ij}^* = 1 - \frac{P(X_i = 0, X_j = 1)}{P(X_i = 0)P(X_j = 1)}$$

$$= 1 - \frac{0.48}{0.96 \times 0.5} = 0$$

implies  $X_i \not\rightarrow X_j$  (14)

$$r_{ji}^* = 1 - \frac{P(X_j = 0, X_i = 1)}{P(X_j = 0)P(X_i = 1)}$$

$$= 1 - \frac{0.02}{0.5 \times 0.04} = 0$$

implies  $X_j \not\rightarrow X_i$ , (15)

Both of (14) and (15) show that no contradiction be leaded.

**Example 4:** the data is the same as Example 2.

In Table 3, from Example 2. we know

that  $\rho_{jk} = \rho_{kj} = 0.798$ ,

and 
$$r_{jk}^* = r_{kj}^* = 1 - \frac{0.05}{0.55 \times 0.45} = \frac{395}{495} > 0.5 \quad (16)$$

implies  $X_j \rightarrow X_k, X_k \rightarrow X_j \Rightarrow X_j \leftrightarrow X_k$  (17)

there are no contradiction to be leaded.

From Example 3 and Example 4, we know that Takeya's IRS algorithm is a valid method to identify the ordering relation between any two items

IV. TRANSITION AND SUBSTITUTION RULES

In both Ordering Theory and Item Relational Structure Theory, the following important issue needs to be considered; whether the transition rule and the substitution rule are existent?

A. Whether the transition rule is existent?

In Correlation Theory, from the formula (18) and (19), we can obtain the Correlation coefficient between any two random variables;

$$\rho_{xy} = \frac{E(X - \mu_x)(Y - \mu_y)}{\sqrt{E(X - \mu_x)^2} \sqrt{E(Y - \mu_y)^2}} = \text{Cos}\theta_{xy} \quad (18)$$

Where  $x = X - \mu_x, y = Y - \mu_y$  (19)

And then we can find that the transition rule is not existent, see also the Figure 1, it shows that the random variable X and Y may be zero correlative even though random variable X and Z are highly correlative with following correlation coefficient value

$$\rho_{xz} = \text{Cos}\theta_{xy} = \text{Cos}\frac{\pi}{4} = \frac{\sqrt{2}}{2} > 0.7, \quad (20)$$

and random variable Z and Y are also highly correlative with following correlation coefficient value

$$\rho_{zy} = \text{Cos}\theta_{zy} = \text{Cos}\frac{\pi}{4} = \frac{\sqrt{2}}{2} > 0.7. \quad (21)$$

Therefore, in both Ordering Theory and Item Relational Structure Theory, the transition rule is also not existent. In other words, the following formula (22) is not always true;

$$X_i \rightarrow X_j, X_j \rightarrow X_k \Rightarrow X_i \rightarrow X_k \quad (22)$$

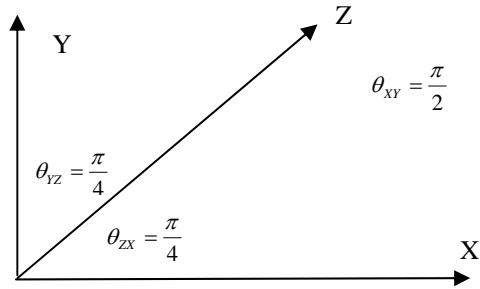


Figure 1 the transition rule is not existent

**B. Whether the substitution rule is existent?**

In both Ordering Theory and Item Relational Structure Theory, the substitution rule is listed as below;

**Substitution rule**

$$X_i \leftrightarrow X_j, X_j \rightarrow X_k \Rightarrow X_i \rightarrow X_k \quad (23)$$

In Takeya's Item Relational Structure Theory, Obviously, if the transition rule is existent, then the substitution rule is also true, however, on the contrary, if the substitution rule is existent, then the substitution rule is not also true, in other words, it means that the substitution rule is stronger than transition rule. In this paper, in author's improved item relational structure theory, the substitution rule must be satisfied, since it is more reasonable for common sense

**V. IMPROVED ITEM RELATIONAL STRUCTURE THEORY ALGORITHM**

Since the threshold limit value of IRS is fixed at 0.5, it lacks of statistical meaning, in addition, this value is not big enough to satisfy the above-mentioned substitution rule, in this paper, an improved item relational structure theory algorithm based on empirical distribution critical value is proposed as below;

**A. Some explained examples about substitution rule with threshold limit value of IRS 0.5 and 0.6**

- (I) If the threshold limit value of IRS is 0.5, let
  - (i)  $r_{jk}^* > 0,5, r_{kj}^* \leq 0,5 \Leftrightarrow I_j \rightarrow_{0.5} I_k$  (24)
  - (ii)  $r_{jk}^* \leq 0,5 \Leftrightarrow I_j \nrightarrow_{0.5} I_k$  (25)
  - (iii)  $r_{jk}^* > 0,5, r_{kj}^* > 0,5 \Leftrightarrow I_j \leftrightarrow_{0.5} I_k$  (26)
  - (iv)  $r_{jk}^* \leq 0,5, \text{ or } r_{kj}^* \leq 0,5 \Leftrightarrow I_j \leftrightarrow_{0.5} I_k$  (27)
- (II) If the threshold limit value of IRS is 0.6, let
  - (i)  $r_{jk}^* > 0,6, r_{kj}^* \leq 0,6 \Leftrightarrow I_j \rightarrow_{0.6} I_k$  (28)
  - (ii)  $r_{jk}^* \leq 0,6 \Leftrightarrow I_j \nrightarrow_{0.6} I_k$  (29)

$$(iii) r_{jk}^* > 0,6, r_{kj}^* > 0,6 \Leftrightarrow I_j \leftrightarrow_{0.6} I_k \quad (30)$$

$$(iv) r_{jk}^* \leq 0,6, \text{ or } r_{kj}^* \leq 0,6 \Leftrightarrow I_j \leftrightarrow_{0.6} I_k \quad (31)$$

The following three cases need to be considered;

$$(i) X_i \leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.5} X_k, X_j \nrightarrow_{0.6} X_k \quad (32)$$

$$(ii) X_i \leftrightarrow_{0.5} X_j, X_i \leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.6} X_k \quad (33)$$

$$(iii) X_i \leftrightarrow_{0.5} X_j, X_i \leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.5} X_k, X_j \nrightarrow_{0.6} X_k \quad (34)$$

**Example 5:**

Suppose the joint and marginal probabilities of item i and j of 122 subjects are listed as Table 4 and the joint and marginal probabilities of item j and k of 122 subjects are listed as Table 5

$$\text{Where } X_i \leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.5} X_k, X_j \nrightarrow_{0.6} X_k \quad (35)$$

Table 4: The joint probabilities of item i and j

		Item j		Total
		$X_j = 1$	$X_j = 0$	
Item i	$X_i = 1$	0.50	0.05	0.55
	$X_i = 0$	0.05	0.40	0.45
Total		0.55	0.45	1

$$r_{ij}^* = r_{ji}^* = \frac{395}{495} > 0.6 \Leftrightarrow I_i \leftrightarrow_{0.6} I_j \quad (36)$$

Table 5: The joint probabilities of item j and k

		Item k		Total
		$X_k = 1$	$X_k = 0$	
Item j	$X_j = 1$	0.26	0.44	0.70
	$X_j = 0$	0.04	0.26	0.30
Total		0.30	0.70	1

$$0.5 < r_{jk}^* = \frac{5}{9} < 0.6 \Leftrightarrow I_j \rightarrow_{0.5} I_k, I_j \nrightarrow_{0.6} I_k \quad (37)$$

We have  $X_i \leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.5} X_k, X_j \nrightarrow_{0.6} X_k$

**Example 6**

Suppose the joint and marginal probabilities of item j and k of 122 subjects are listed as Table 6 and the joint and marginal probabilities of item j and k of 122 subjects are listed as Table 7

$$\text{where } X_i \leftrightarrow_{0.5} X_j, X_i \leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.6} X_k \quad (38)$$

Table 6: The joint probabilities of item i and j

		Item j		Total
		$X_j = 1$	$X_j = 0$	
Item i	$X_i = 1$	0.44	0.11	0.55
	$X_i = 0$	0.11	0.34	0.45
Total		0.55	0.45	1

$$0.5 < \gamma_{ij}^* = \gamma_{ji}^* = \frac{5}{9} < 0.6 \Rightarrow X_i \leftrightarrow_{0.5} X_j, X_i \not\leftrightarrow_{0.6} X_j \quad (39)$$

Table 7 The joint probabilities of item j and k

		Item k		Total
		$X_k = 1$	$X_k = 0$	
Item j	$X_j = 1$	0.32	0.38	0.70
	$X_j = 0$	0.03	0.27	0.30
Total		0.35	0.65	1

$$r_{jk}^* = \frac{5}{7} > 0.6, r_{kj}^* = \frac{11}{87} < 0.5 \Rightarrow X_j \rightarrow_{0.6} X_k \quad (40)$$

We have  $X_i \leftrightarrow_{0.5} X_j, X_i \not\leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.6} X_k$

**Example 7**

Suppose the joint and marginal probabilities of item j and k of 122 subjects are listed as Table 8 and the joint and marginal probabilities of item j and k of 122 subjects are listed as Table 9

where

$$X_i \leftrightarrow_{0.5} X_j, X_i \not\leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.5} X_k, X_j \not\rightarrow_{0.6} X_k \quad (41)$$

Table 8 The joint probabilities of item i and j

		Item j		Total
		$X_j = 1$	$X_j = 0$	
Item i	$X_i = 1$	0.42	0.12	0.54
	$X_i = 0$	0.12	0.34	0.46
Total		0.54	0.46	1

$$0.5 < r_{ij}^* = r_{ji}^* = \frac{321}{621} < 0.6 \Rightarrow X_i \leftrightarrow_{0.5} X_j, X_i \not\leftrightarrow_{0.6} X_j \quad (42)$$

Table 9 The joint probabilities of item j and k

		Item k		Total
		$X_k = 1$	$X_k = 0$	
Item j	$X_j = 1$	0.26	0.44	0.70
	$X_j = 0$	0.04	0.26	0.30
Total		0.30	0.70	1

$$0.5 < \gamma_{jk}^* = \frac{143}{243} < 0.6, \gamma_{kj}^* = \frac{157}{657} < 0.5 \Rightarrow X_j \rightarrow_{0.5} X_k, X_j \not\rightarrow_{0.6} X_k \quad (43)$$

We have  $X_i \leftrightarrow_{0.5} X_j, X_i \not\leftrightarrow_{0.6} X_j, X_j \rightarrow_{0.6} X_k$

From the above three examples, we know that the threshold limit value of IRS 0.6 is better than 0.5, in other words, Takeya's fixed threshold limit value of IRS is worse than 0.6, in addition, it lacks of statistical meaning.

*B. Improved Item Relational Structure Algorithm based on empirical distribution critical value*

let n be number of items, m be number of examinees, therefore, the number of all ordering index  $r_{jk}^*$  is  $n(n-1)$ , then we can obtain a distribution of all ordering index  $r_{jk}^*$ ,

and let the threshold limit value of IRS be defined as

$$r_c = \arg \left[ 1 - \int_{-\infty}^x f(r_{jk}^*) dr_{jk}^* = 0.05 \right] \quad (44)$$

where  $f(r_{jk}^*)$  is the probability density function of random variable  $r_{jk}^*$ .

Test  $H_0 : \rho_c \leq 0$  vs  $H_1 : \rho_c > 0$ , at  $\alpha = 0.05$ , using the statistic [5-6]

$$t_c = \frac{r_c}{\sqrt{\frac{1 - [r_c]^2}{m - 2}}} \sim t_{(m-2)} \quad (45)$$

where m is the number of sample size.

If  $t_c > t_{1-0.05}(m-2)$ , then reject  $H_0$ , in other words  $r_c$  can be used as a valid threshold limit value of IRS, therefore an improved IRS algorithm based on empirical distribution critical value is proposed. Here we define

$$X_j \rightarrow X_k, \text{ if and only if } r_{jk}^* > r_c, r_{kj}^* \leq r_c \quad (46)$$

$$X_k \rightarrow X_j, \text{ if and only if } r_{kj}^* > r_c, r_{jk}^* \leq r_c \quad (47)$$

$$X_j \leftrightarrow X_k \text{ if and only if } r_{jk}^* > r_c, r_{kj}^* > r_c. \quad (48)$$

VI. MATLAB PROGRAM OF NEW METHOD

The Matlab program of item relational structure theory algorithm based on empirical distribution critical value is listed as bellow;

```
clear all
%%%-----load data-----
disp('data loading...');
data = xlsread('test8.xls');
disp('End of data loading...');
%%%----- End of data loading -----
[m,n]=size(data);
IRS=zeros(n);
Total=zeros(1,n*(n-1));
k=1;
for i=1:n
    for j=1:n
        if i == j
            IRS(i,j)=0;
        else
            IRS(i,j)=irs(data(:,i),data(:,j));
            Total(k)=IRS(i,j);
            k=k+1;
        end
    end
end
Total_sorted=sort(Total);
value=Total_sorted(round(length(Total)*.95))

Tradition=IRS;
New_method=IRS;

Tradition(find(Tradition(:,:)>=0.5))=1;
Tradition(find(Tradition(:,:<0.5))=0;

New_method(find(New_method(:,:)>=value))=1;
New_method(find(New_method(:,:<value))=0;

disp(['Number of total items ' num2str(n) ' ° ']);
questions=[];
question_no = input('Please input the number of your reserved item or press "Enter" to stop item-selecting :');
while ( question_no >= 1 & question no <= n )
    questions = cat(2,questions,question_no);
```

```
question_no = input(' Please input the number of your reserved item or press "Enter" to stop item-selecting :');
end
if length(questions)==0
    disp(['you did not select any item.']);
    break
else
    disp(['the number of your selected item is item ' num2str(questions) ' ']);
end
disp(['Please press any key to continue ']);
pause
questions=unique(questions);
question_del=n:-1:1;
D=sort(-questions);
for d = -D
    index=find(question_del==d);
    question_del(index)=[];
end
for d = question_del
    Tradition(:,d)=[];
    Tradition(d,:)=[];
    New_method(:,d)=[];
    New_method(d,:)=[];
end

Tradition
New_method
```

VII. EXPERIMENTS AND RESULTS

For comparing the performances of Takeya’s IRS algorithm, and our new method, a fraction addition test with 30 items was administrated to 122 5th grade students in Taiwan. For convenience, only the item ordering structures of 7 items of all students are shown in Table 7.

Table 7. Items used in IRS and their chacteristic

Item	Denominators are not the same.	The answer is greater than 1.
(1) $\frac{2}{4} + \frac{1}{4}$		
(2) $\frac{4}{7} + \frac{6}{7}$		X
(7) $\frac{3}{7} + \frac{1}{5}$	X	
(10) $\frac{3}{4} + \frac{5}{7}$	X	X
(17) $\frac{1}{7} + \frac{3}{7} + \frac{2}{7}$		
(20) $\frac{1}{4} + \frac{2}{7} + \frac{1}{7}$	X	
(23) $\frac{1}{8} + \frac{2}{7} + \frac{1}{4}$	X	

Table 7. shows that the following facts;

- (i) the item whose denominators are not the same is more difficulty than the item whose denominators are the same
- (ii) the item whose answer is greater than 1 is more difficulty than the item whose answer is not greater than 1
- (iii) the item with 3 fractions is more difficulty than the item with 2 fractions

Then, in general, we know that basic item ordering relations may be listed as below

$$I_1 \rightarrow I_2, I_1 \rightarrow I_7, I_1 \rightarrow I_{10}, I_1 \rightarrow I_{17}, I_1 \rightarrow I_{20}, I_1 \rightarrow X_{23} \quad (49)$$

$$I_2 \rightarrow I_7, I_2 \rightarrow I_{10}, I_2 \rightarrow I_{17}, I_2 \rightarrow I_{20}, I_2 \rightarrow X_{23} \quad (50)$$

$$I_7 \rightarrow I_{10}, I_7 \rightarrow I_{17}, I_7 \rightarrow I_{20}, I_7 \rightarrow X_{23} \quad (51)$$

$$I_{10} \rightarrow I_{17}, I_{10} \rightarrow I_{20}, I_{10} \rightarrow X_{23} \quad (52)$$

$$I_{17} \rightarrow I_{20}, I_{17} \rightarrow X_{23} \quad (53)$$

$$I_{20} \rightarrow X_{23} \quad (54)$$

The experimental result gives the Figure 2 and 3, they are item ordering structures generated by Takeya's IRS algorithm with threshold 0.5 and our new method with the empirical distribution critical value threshold 0.628 respectively. This result shows that the item ordering structures constructed by Takeya's IRS algorithm with threshold 0.5 is not reasonable. Since in Figure 1(a), we can find that,

$$X_{10} \leftrightarrow X_7, X_7 \rightarrow X_{20} \Rightarrow X_{10} \rightarrow X_{20}, \quad (55)$$

but  $X_{10} \rightarrow X_{20}$  does not exist, it leads a contradiction, and our new method is reasonable.

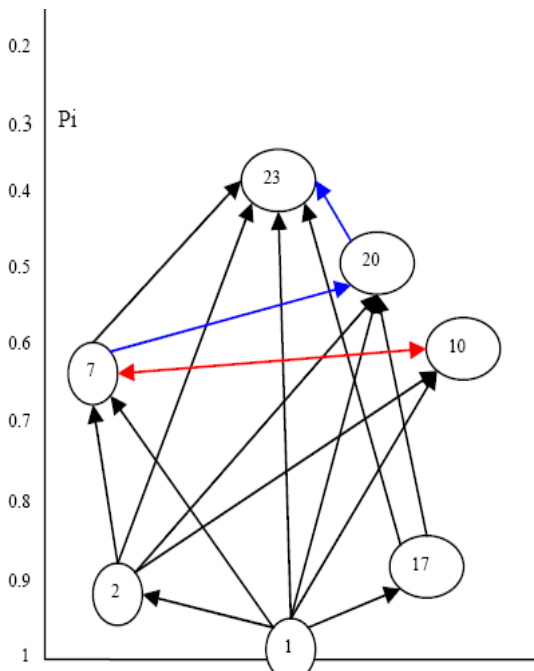


Figure 2. IRS of 122 examinees generated by Takeya's threshold 0.5

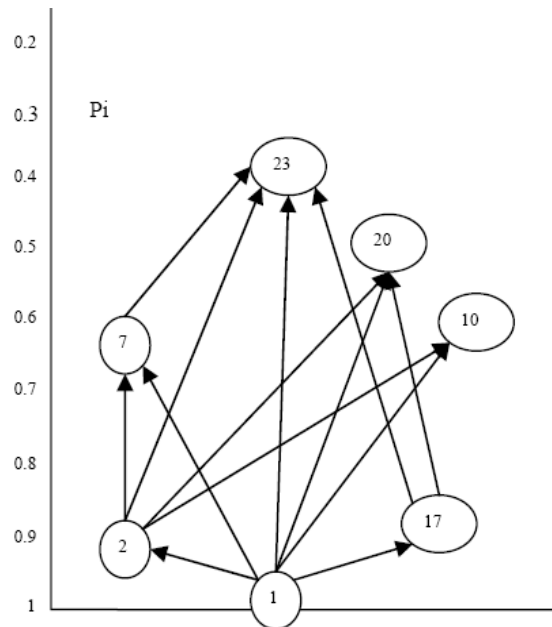


Figure 3. IRS of 122 examinees generated by new method threshold 0,628

### VIII. CONCLUSION

There are two well-known ordering item algorithms based on the testing performance of a group of subjects, one is Bart et al's OT algorithm, and the other is Takeya's IRS algorithm, the former do not consider the correlation relation as the later. However, the threshold limit values of both of them are fixed values, lacking of statistical meaning. In this paper, the authors consider to improve the threshold limit value by using the empirical distribution critical value of all the values of the relational structure indices between any two items, it is more valid than Takeya's threshold 0.5 for comparing the ordering relation of any two items. A computer program is developed for the proposed method. A Mathematics example is also provided in this paper to illustrate the advantages of the proposed methods

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Hsiang-Chuan Liu received the Ph.D. degree in Statistics from National Taiwan University, Taiwan. He is a professor at the Department of Bioinformatics and Medical Informatics and Department of Psychology, Asia University, Taiwan since August 2001, and a professor at the

Graduate Institute of Acupuncture Science, China Medical University, and also an honored professor at the Graduate Institute of Educational Measurement and Statistics, National Taichung University, Taiwan. He was the President of National Taichung University, Taiwan from 1993 to 2000. Dr. Liu is a member of IEEE since 2007. He has funded research and published articles in the areas of Biostatistics, Bioinformatics, Fuzzy Theory, Educational Measurement, and E-Learning.



Shih-Neng Wu is a Ph. D. student of Department of Computer Science and Information Engineering, Asia University, Taiwan. His studies focus in Educational Measurement and Statistics, Item Relational Structure and Ordering Theory, Fuzzy Measure Theory and Intuitionistic Fuzzy Set Theory, etc.



Chin-Chun Chen received the Ph.D. degree in Educational Measurement and Statistics from National Taichung University. He is an Associate Professor and the President of Min-Hwei College, Taiwan. His studies focus in Educational Measurement and Statistics, Fuzzy Measure Theory