A full-Newton step interior-point algorithm based on modified-Newton direction

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***Abstract*— By a modification of the classic-Newton direction in scaled version for linear optimization, we give a new interior-point algorithm based on a very simple function. The algorithm uses full modified-Newton step, thus no need to perform line search. In the processing of the algorithm, the simple function is used to control the searching direction and measure the proximity of iterates to the central path. Moreover, the modified-Newton step used in the algorithm has local quadratic convergence property according to the proximity function. The iteration complexity is derived, and which is the best-known.**

***Index Terms*— complexity analysis, linear optimization, interior-point algorithm, modified-Newton direction.**

I. INTRODUCTION

Interior-point methods (IPMs) for solving linear optimization (LO) problems were initiated by Karmarkar [1]. They not only have polynomial complexity but are also highly efficient in practice. For a survey we refer to recent books on the subject [2], [3].

Many IPMs use the classic-Newton direction, Some algorithms explicitly use this direction as they force the iterate points to follow the central path. Even for many algorithms that do not use this direction directly in the algorithm statements, the classic-Newton direction is used as the basis for deriving the new direction [2], [4]–[7].

Because the importance of the classic-Newton direction in the designs and analysis of IPMs, we study the Newton direction.

Recently, a new class of primal-dual IPMs was introduced. These methods do not use the classic-Newton direction. Instead they use a direction that can be characterized as a steepest descent direction (in a scaled space) for some kinds of barrier function, see Bai, Peng, etc., [4]–[11]. Any such barrier function is determined by a simple univariate function, called its kernel function. It should be noted that all the aforementioned kernel function based methods use the damped step size.

A kernel function is a univariate strictly convex function which is defined for all positive reals t and which is minimal at $t = 1$ whereas the minimal value equals 0. To use the kernel function in the designing of algorithm, many restrictions have been set on, for example self-regular, exponential convexity, etc. These restrictions make the expression of kernel function very complex. The interested reader may refer [4]–[11] for further understanding.

Considering that, for most of the IPMs, the iterates generated by the algorithm lie in some neighborhood of central path. The central path can be interpreted as a sequence of minimizer points, which attain their minimal value zero at e. In this paper, we want to define such a neighborhood, which not only contains all the iterates but also can be used as feasibility testification for searching direction. For this, we introduce the following simple function

$$\psi_s(t) = (1-t)^2. \tag{1}$$

Using the simple function on the coordinates, we construct a separable function

$$\Psi_s(v) = \sum_{i=1}^{n} \psi_s(v_i) = \sum_{i=1}^{n} (1-v_i)^2. \tag{2}$$

As easily be verified, letting $\Psi(v) < 1$, all the coordinates of v must positive, which means the positive constraints for variables hold. Thus the separable function has some kind of barrier property for restricting the iterates be infeasible. As an application, we present an IPM based on this simple function, and the complexity analysis shows that the induced algorithm enjoys the best-known iteration bound for LO.

The structure of the paper is as follows. In Section II, we introduce how the simple function can be used to generate the search direction for IPM. Section III, we investigate some useful properties of the simple kernel function. These properties will be used as tools in the analysis of the IPM. In Section IV, we present a full modified-Newton step IPM based on the simple function. The full modified-Newton step used in the algorithm has the quadratic convergence property in some neighborhood of central-path. Furthermore, we derive the complexity for the algorithm and obtain the best-known result for LO. Some conclusions and remarks are given in Section V.

Some notation used throughout the paper is as follows. The 2-norm and the infinity norm are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. If $x, s \in R^n$, then xs denotes the componentwise (or Hadamard) product of the vectors

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x and s. $\min(v)$ and $\max(v)$ denote the minimize and maximize components of the vector v, respectively.

II. The modified-Newton direction for LO

In this paper, we consider the LO problem

$$(P) \qquad \min\{c^T x : Ax = b, x \geq 0\},$$

where $A \in R^{m\times n}$, $\text{rank}(A) = m,\ b \in R^m,\ c \in R^n$, and its dual problem

$$(D) \qquad \max\{b^T y : A^T y + s = c, s \geq 0\}.$$

Without loss of generality [2], we assume that (P) and (D) satisfy the interior-point condition (IPC), i.e., there exist $x^0,\ y^0$, and s^0 such that

$$Ax^0 = b,\ x^0 > 0,\ A^T y^0 + s^0 = c,\ s^0 > 0.$$

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the nonlinear system as follows,

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ xs & = & 0. & \end{array} \tag{3}$$

The third equation in system (3) is called the complementarity condition.

A. The central path

The basic idea underlying primal-dual IPMs is to replace the complementarity condition by the nonlinear equation $xs = \mu e$, with parameter $\mu > 0$ and with $e = (1, \ldots, 1)^T$. The system (3) now becomes:

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ xs & = & \mu e. & \end{array} \tag{4}$$

The existence of a unique solution to the above system is well-known [12]. We denote the unique solution of system (4) by $(x(\mu), y(\mu), s(\mu))$ for each $\mu > 0$. $x(\mu)$ is called the μ-center of (P) and $(y(\mu), s(\mu))$ is the μ-center of (D). The set of μ-centers (with $\mu > 0$) defines a homotopy path, which is called the central path of (P) and (D) [13]. If μ goes to zero then the limit of the central path exists. This limit satisfies the complementarity condition, and hence yields optimal solutions for (P) and (D) [2].

B. The classic-Newton direction

Primal-dual IPMs follow the central path approximately. Let us briefly indicate how this goes. Without loss of generality we assume that $(x(\mu), y(\mu), s(\mu))$ is known for some positive μ [2], (which means that we have a primal feasible $x(\mu) > 0$ and dual feasible $y(\mu)$ and $s(\mu) > 0$).

In feasible IPM, we are given a positive feasible pair (x, s), and some $\mu > 0$. Our aim is to define search directions $(\triangle x, \triangle s)$ that move in the direction of the μ-center $(x(\mu), s(\mu))$. In fact, we want the new iterates $x + \triangle x$, $s + \triangle s$ to satisfy system (4) and be positive with respect to μ. After substitution this yields the following conditions on $(\triangle x, \triangle s)$

$$\begin{array}{rcll} A(x + \triangle x) & = & b, & x + \triangle x > 0, \\ A^T(y + \triangle y) + (s + \triangle s) & = & c, & s + \triangle s > 0, \\ (x + \triangle x)(s + \triangle s) & = & \mu e. & \end{array}$$

If we neglect for the moment the inequality constraints, then, since $Ax = b$ and $A^T y + s = c$, this system can be rewritten as follows

$$\begin{array}{rcl} A\triangle x & = & 0, \\ A^T \triangle y + \triangle s & = & 0, \\ x\triangle s + s\triangle x & = & \mu e - xs. \end{array} \tag{5}$$

Since A has full row rank, the above system uniquely defines a search direction $(\triangle x, \triangle y, \triangle s)$ for any $x > 0$ and $s > 0$ [2], and this is the so-called classic-Newton direction. The third equation is called the centering equation.

C. Scaling

We follow [5], [7], [9] and reformulate the above approach by defining the same search direction in a different way.

To make this clear, in what follows, we associate to any triple (x, s, μ), with $x > 0$, $s > 0$ and $\mu > 0$, the vector

$$v = \sqrt{\frac{xs}{\mu}}.$$

Note that if x is primal feasible and s is dual feasible then the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = e$. Introducing the notations

$$\bar{A} = AV^{-1}X \text{ and } V := \text{diag}(v),\ X := \text{diag}(x), \tag{6}$$

and defining the scaled search directions d_x and d_s according to

$$d_x := \frac{v\triangle x}{x} \text{ and } d_s := \frac{v\triangle s}{s}. \tag{7}$$

Using this notation, the system (5) can be rewritten as

$$\begin{array}{rcl} \bar{A}d_x & = & 0, \\ \frac{1}{\mu}\bar{A}^T\triangle y + d_s & = & 0, \\ d_x + d_s & = & v^{-1} - v. \end{array} \tag{8}$$

Note that d_x and d_s are orthogonal vectors, since the vector d_x belongs to the null space and d_s to the row space of the matrix $\bar{A}$. Hence we will have $d_x = d_s = 0$ if and only if $v^{-1} - v = 0$, which is equivalent to $v = e$. We conclude that $d_x = d_s = 0$ holds if and only if the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$.

D. The modified-Newton direction

In the scaled centering equation we replace $v^{-1} - v$ by the negative gradient of the convex function $\Psi_s(v)$. Thus the new search direction is obtained by solving the system

$$\begin{array}{rcl} \bar{A}d_x & = & 0, \\ \frac{1}{\mu}\bar{A}^T\triangle y + d_s & = & 0, \\ d_x + d_s & = & -\nabla\Psi_s(v), \end{array} \tag{9}$$

for d_x, $\triangle y$ and d_s, and then computing $\triangle x$ and $\triangle s$ from

$$\triangle x = \frac{xd_x}{v},\ x = \frac{sd_s}{v},$$

according to (7).

III. The simple function

In the following, we give some basic properties of the simple function $\psi_s(t)$. After some basic calculations, one has

$$\psi_s'(t) = -2(1-t) \text{ and } \psi_s''(t) = 2. \tag{10}$$

It easily verify that $\psi_s(1) = \psi_s'(1) = 0$ and $\psi_s(t)$ is strictly convex.

Lemma 1: One has

$$\psi_s(t) = \frac{1}{4}\psi_s'(t)^2 \text{ and } -t\psi_s'(t) = 2\left(t - t^2\right).$$

Proof: The results are immediately from (10). ■

The scaled function induced by $\psi_s(t)$ is given as (2). Since $v > 0$ such that $\Psi_s(v)$ is minimal at $v = e$ and $\Psi_s(e) = 0$, thus $\Psi_s(v)$ can also be used as proximity function to show how close the iterates is to the central path.

Corollary 1: One has

$$-v\nabla\Psi_s(v) = 2\left(v - v^2\right).$$

Proof: It is the direction consequence of the second part of Lemma 1 in vector form. ■

The following lemma estimates the lower bound for the components of v in terms of $\Psi_s(v)$.

Lemma 2: One has

$$\min(v) \geq 1 - \sqrt{\Psi_s(v)}.$$

Proof: Using (2) and Lemma 1, one has

$$\Psi_s(v) = \sum_{i=1}^{n}\psi_s(v_i) = \frac{1}{4}\sum_{i=1}^{n}\psi_s'(v_i)^2 = \frac{1}{4}\|\nabla\Psi_s(v)\|^2.$$

Thus

$$2\sqrt{\Psi_s(v)} = \|\nabla\Psi_s(v)\| \geq |\psi_s'(\min(v))| \geq -\psi_s'(\min(v)).$$

Using (10), one has

$$\psi_s'(\min(v)) = -2(1 - \min(v)),$$

thus

$$\sqrt{\Psi_s(v)} \geq 1 - \min(v).$$

This completes the proof. ■

We give an upper bound for the 2-norm of vector v in terms of $\Psi_s(v)$.

Lemma 3: One has

$$\|v\| \leq \sqrt{n} + \sqrt{\Psi_s(v)}.$$

Proof: Using (2), one has

$$\Psi_s(v) = \sum_{i=1}^{n}(1-v_i)^2 = \|v\|^2 - 2e^Tv + n \geq (\|v\| - \|e\|)^2.$$

This implies

$$\|v\| \leq \|e\| + \sqrt{\Psi_s(v)} = \sqrt{n} + \sqrt{\Psi_s(v)},$$

which completes the proof. ■

IV. Full modified-Newton step IPM

It is assumed that we are given a positive primal-dual pair $\left(x^0, s^0\right) > 0$ and $\mu^0 > 0$ such that $\left(x^0, s^0\right)$ is close to the μ^0-center in the sense of the proximity function $\Psi_s(v^0) \leq \tau$, where $v^0 = \sqrt{\frac{x^0s^0}{\mu^0}}$.

Generic IPM with full modified-Newton steps

Input: Accuracy parameter $\epsilon > 0$;
barrier update parameter $\theta, 0 < \theta < 1$;
threshold parameter $\tau, 0 < \tau < 1$;
a strictly feasible pair $\left(x^0, s^0\right)$ with $\mu^0 > 0$ such that $\Psi_s(v^0) \leq \tau$.

begin:
$x := x^0;\ s := s^0;\ \mu := \mu^0.$
while $x^Ts \geq \epsilon$
solve (9) and obtain $(\triangle x, \triangle s)$, let
$x := x + \triangle x;$
$s := s + \triangle s;$
μ-update: $\mu := (1-\theta)\mu;$
endwhile
end

Figure 1.

A crucial question is, of course, how to choose the parameters that control the algorithm, i.e., the threshold parameter τ, the barrier update parameter θ, so as to minimize the iteration complexity.

A. Some useful tools

We first investigate the proximity function $\Psi_s(v)$ in details. It follows from system (2), (9) and Lemma 1 that

$$\Psi_s(v) = \frac{1}{4}\|d_x + d_s\|^2. \tag{11}$$

We denote

$$p_v = d_x + d_s \text{ and } q_v = d_x - d_s. \tag{12}$$

Thus one has

$$\Psi_s(v) = \frac{1}{4}\|p_v\|^2. \tag{13}$$

Lemma 4: Let p_v and q_v defined as (12), one has

$$\|p_v\| = \|q_v\|.$$

Proof: Using the orthogonality property of d_x and d_s and (12), we have

$$\begin{aligned}\|p_v\|^2 &= e^T(d_x + d_s)^2 \\ &= \|d_x\|^2 + 2(d_x)^T d_s + \|d_s\|^2 \\ &= \|d_x\|^2 - 2(d_x)^T d_s + \|d_s\|^2 \\ &= e^T(d_x - d_s)^2 \\ &= \|q_v\|^2,\end{aligned}$$

which completes the proof. ■

B. Feasibility of a full modified-Newton step

We denote the result of the full modified-Newton step given by system (9) at (x, y, s) by (x^+, y^+, s^+), ie.,

$$x^+ := x + \triangle x, \ , y^+ := y + \triangle y, \ s^+ := s + \triangle x.$$

Using (7), one has

$$x^+ = x + \triangle x = x\left(e + \frac{d_x}{v}\right) = \frac{x}{v}\left(v + d_x\right),$$

and

$$s^+ = s + \triangle s = s\left(e + \frac{d_s}{v}\right) = \frac{s}{v}\left(v + d_s\right).$$

We associate to any triple (x^+, s^+, μ), with $x^+ > 0$, $s^+ > 0$ and $\mu > 0$, the vector

$$v^+ = \sqrt{\frac{x^+ s^+}{\mu}}. \tag{14}$$

Remember that, by the third equation of system (9), we have $d_x + d_s = -\nabla\Psi_s(v)$. It follows from Corollary 1 that

$$\begin{aligned}\left(v^+\right)^2 &= \frac{x^+ s^+}{\mu} \\ &= \left(v + d_x\right)\left(v + d_s\right) \\ &= v^2 + v(d_s + d_x) + d_x d_s \\ &= e - (e - v)^2 + d_x d_s. \end{aligned} \tag{15}$$

Thus, the condition for feasibility step follows.

Lemma 5: The new iterates (x^+, y^+, s^+) are strictly feasible if

$$\Psi_s(v) < 1.$$

Proof: For the proof we introduce a step length $\alpha \in [0, 1]$, and define

$$x(\alpha) = x + \alpha\triangle x, \ y(\alpha) = y + \alpha\triangle y, \ s(\alpha) = s + \alpha\triangle s.$$

We then have $x(0) = x$, $x(1) = x^+$ and similarly $s(0) = s$, $s(1) = s^+$. Hence, we have

$$x^0 s^0 = xs > 0.$$

We write

$$\begin{aligned} x(\alpha)s(\alpha) &= (x + \alpha\triangle x)(s + \alpha\triangle s) \\ &= xs + \alpha(x\triangle s + s\triangle x) + \alpha^2\triangle x\triangle s. \end{aligned}$$

Thus

$$\frac{x(\alpha)s(\alpha)}{\mu} = v^2 + \alpha v\left(d_x + d_s\right) + \alpha^2 d_x d_s.$$

Using Corollary 1, ie., $v(d_x + d_s) = 2\left(v - v^2\right)$, we obtain

$$\frac{x(\alpha)s(\alpha)}{\mu} = (1 - \alpha)v^2 + \alpha\left(e - (v - e)^2\right) + \alpha^2 d_x d_s.$$

Therefore, the components of $x(\alpha)s(\alpha)$ will be positive if

$$(1 - \alpha)v^2 + \alpha(e - (v - e)^2) + \alpha^2 d_x d_s > 0. \tag{16}$$

Since $\alpha \in [0, 1]$, we have that $(1 - \alpha)v^2 \geq 0$. Thus the inequality (16) is true for

$$e - (v - e)^2 + \alpha d_x d_s > 0. \tag{17}$$

Using (12), and by (10) and system (9), one has

$$p_v = 2(e - v) \text{ and } p_v^2 - q_v^2 = 4 d_x d_s. \tag{18}$$

thus the inequality (17) can be interpreted as

$$e - \frac{p_v^2}{4} + \alpha\frac{p_v^2 - q_v^2}{4} > 0,$$

which means

$$(1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4} < e.$$

Thus one concludes that the components of $x(\alpha)s(\alpha)$ will be positive if

$$\max\left((1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right) < 1. \tag{19}$$

Since

$$\begin{aligned}\left\|(1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\| &\leq (1 - \alpha)\frac{\left\|p_v^2\right\|}{4} + \alpha\frac{\left\|q_v^2\right\|}{4} \\ &\leq (1 - \alpha)\frac{\left\|p_v\right\|^2}{4} + \alpha\frac{\left\|q_v\right\|^2}{4} \\ &= \frac{\left\|p_v\right\|^2}{4}, \end{aligned}$$

the last equation follows from Lemma 4. Using (13), we have

$$\left\|(1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\| \leq \Psi_s(v).$$

Assuming that $\Psi_s(v) < 1$, one has

$$\left\|(1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\| \leq 1.$$

Since

$$\max\left((1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right) \leq \left\|(1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|,$$

we arrive that

$$\max\left((1 - \alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right) < 1$$

certainly holds. Hence, the the components of $x(\alpha)s(\alpha)$ are positive for $\alpha \in [0, 1]$. Since $x(0)$ and $s(0)$ are positive and since $x(\alpha)$ and $s(\alpha)$ depend continuously on α, it follows that $x(1)$ and $s(1)$ are also positive. This completes the proof. ■

C. Duality gap

In general the new iterates x^+ and s^+ do not coincide with μ-centers. But we have the surprising property that the duality gap less than the value at the μ-centers, where the duality gap equals $n\mu$.

Lemma 6: The duality gap after a full modified-Newton step satisfies

$$\left(x^+\right)^T s^+ \leq \mu n.$$

Proof: It follows from (15) and by (18), we have

$$\begin{aligned}(v^+)^2 &= e - (v - e)^2 + d_x d_s \\ &= e - \frac{p_v^2}{4} + \frac{p_v^2 - q_v^2}{4} \\ &= e - \frac{q_v^2}{4}. \end{aligned} \tag{20}$$

Using (14) gives

$$\begin{aligned}\left(x^+\right)^T s^+ &= \mu e^T \left(v^+\right)^2 = \mu e^T \left(e - \frac{q_v^2}{4}\right) \\ &= \mu\left(n - \frac{\|q_v\|^2}{4}\right) \leq \mu n,\end{aligned}$$

which completes the proof. ■

D. Quadratic convergence

To prove the effect of a full modified-Newton step on proximity function, the following lemma is needed.

Lemma 7: Let $\Psi_s(v) < 1$. Then one has

$$\min(v^+) \geq \sqrt{1 - \Psi_s(v)}.$$

Proof: Using (20), one has

$$\min(v^+)^2 = \min\left(e - \frac{q_v^2}{4}\right) \geq 1 - \frac{\max\left(q_v^2\right)}{4}.$$

Since, by (13), one has

$$\Psi_s(v) = \frac{\|q_v\|^2}{4} \geq \frac{\max\left(q_v^2\right)}{4},$$

the inequality

$$\min(v^+)^2 \geq 1 - \Psi_s(v)$$

certainly holds. This completes the proof. ■

Theorem 8: If $\Psi_s(v) < 1$ and v^+ is defined as (14), then we have

$$\Psi_s(v^+) < \Psi_s(v)^2.$$

Proof: Using (20), we have

$$\begin{aligned}\sum_{i=1}^n \left(1 - v_i^+\right)^2 &= \sum_{i=1}^n \frac{\left(1 - \left(v_i^+\right)^2\right)^2}{\left(1 + v_i^+\right)^2} \\ &\leq \frac{\left\|e - \left(v^+\right)^2\right\|^2}{\left(1 + \min(v^+)\right)^2} \\ &\leq \frac{\left(\frac{\|q_v^2\|}{4}\right)^2}{\left(1 + \sqrt{1 - \Psi_s(v)}\right)^2} \\ &\leq \frac{\left(\frac{\|q_v\|^2}{4}\right)^2}{\left(1 + \sqrt{1 - \Psi_s(v)}\right)^2}.\end{aligned}$$

Thus, by (13), we have

$$\begin{aligned}\Psi_s(v^+) &= \sum_{i=1}^n \left(1 - v_i^+\right)^2 \\ &\leq \frac{\Psi_s(v)^2}{\left(1 + \sqrt{1 - \Psi_s(v)}\right)^2} \\ &< \Psi_s(v)^2,\end{aligned}$$

the results follows. ■

Remark 1: Theorem 8 implies that after a full modified-Newton step the proximity to the μ-center is small than the square of the proximity before the full modified-Newton step. In other words, full modified-Newton step is quadratically convergent. Moreover, the theorem defines a neighborhood of the μ-center where the quadratic convergence occurs, namely

$$\Psi_s(v) < 1.$$

This result is extremely important. It implies that when the present iterate is close to the μ-center, then only a small number of full modified-Newton steps brings us very close to the μ-center.

E. The effect of a full modified-Newton step and parameter update on proximity function

In the following, we focus on the decrease of proximity function just before and after a μ-update. Denote $\Psi_s\left(x^+, s^+; \mu^+\right)$ as the proximity after a full modified-Newton step and an μ-update, ie.,

$$\Psi_s(x^+, s^+; \mu^+) = \Psi_s\left(\sqrt{\frac{x^+ s^+}{\mu^+}}\right) = \Psi_s\left(\frac{v^+}{\sqrt{1-\theta}}\right),$$

where v^+ is defined as (14).

Theorem 9: If $\Psi_s(v) \leq 1$, then

$$\Psi_s\left(x^+, s^+; \mu^+\right) \leq \frac{\left(\theta\sqrt{n} + \Psi_s(v)\right)^2}{\left(1 - \theta + \sqrt{1-\theta}\sqrt{1 - \Psi_s(v)}\right)^2}.$$

Proof:

It follows from (20) and Lemma 7, that we have

$$\begin{aligned}\sum_{i=1}^n \left(\sqrt{1-\theta} - v_i^+\right)^2 &= \sum_{i=1}^n \frac{\left(1 - \theta - \left(v_i^+\right)^2\right)^2}{\left(\sqrt{1-\theta} + v_i^+\right)^2} \\ &\leq \frac{\left\|-\theta e + e - \left(v^+\right)^2\right\|^2}{\left(\sqrt{1-\theta} + \min(v^+)\right)^2} \\ &\leq \frac{\left(\theta\sqrt{n} + \frac{\|q_v^2\|}{4}\right)^2}{\left(\sqrt{1-\theta} + \sqrt{1 - \Psi_s(v)}\right)^2} \\ &\leq \frac{\left(\theta\sqrt{n} + \frac{\|q_v\|^2}{4}\right)^2}{\left(\sqrt{1-\theta} + \sqrt{1 - \Psi_s(v)}\right)^2}.\end{aligned}$$

Using (13), we have

$$\sum_{i=1}^n \left(\sqrt{1-\theta} - v_i^+\right)^2 \leq \frac{\left(\theta\sqrt{n} + \Psi_s(v)\right)^2}{\left(\sqrt{1-\theta} + \sqrt{1 - \Psi_s(v)}\right)^2}.$$

Since

$$\begin{aligned}\Psi_s\left(\frac{v^+}{\sqrt{1-\theta}}\right) &= \sum_{i=1}^n \left(1 - \frac{v_i^+}{\sqrt{1-\theta}}\right)^2 \\ &= \frac{1}{1-\theta}\sum_{i=1}^n \left(\sqrt{1-\theta} - v_i^+\right)^2,\end{aligned}$$

one concludes that

$$\Psi_s\left(x^+, s^+; \mu^+\right) \leq \frac{\left(\theta\sqrt{n} + \Psi_s(v)\right)^2}{\left(1 - \theta + \sqrt{1-\theta}\sqrt{1 - \Psi_s(v)}\right)^2}.$$

This completes the proof. ■

F. Fixing the parameter

We want to find a threshold τ and an update parameter θ, which at the start of iterate satisfies $\Psi_s(v) \leq \tau$. After a full modified-Newton step and an μ-update, the property $\Psi_s\left(x^+, s^+; \mu^+\right) \leq \tau$ is maintained. In this case, by Theorem 9, it suffices for

$$\frac{\left(\theta\sqrt{n} + \Psi_s(v)\right)^2}{\left(1-\theta+\sqrt{1-\theta}\sqrt{1-\Psi_s(v)}\right)^2} \leq \tau.$$

The left side of the above inequality is monotone increasing according to $\Psi_s(v)$, which means that

$$\begin{aligned} & \frac{\left(\theta\sqrt{n} + \Psi_s(v)\right)^2}{\left(1-\theta+\sqrt{1-\theta}\sqrt{1-\Psi_s(v)}\right)^2} \\ \leq & \frac{\left(\theta\sqrt{n} + \tau\right)^2}{\left(1-\theta+\sqrt{1-\theta}\sqrt{1-\tau}\right)^2}. \end{aligned}$$

Thus the $\Psi_s\left(x^+, s^+; \mu^+\right) \leq \tau$ suffices for

$$\frac{\left(\theta\sqrt{n} + \tau\right)^2}{\left(1-\theta+\sqrt{1-\theta}\sqrt{1-\tau}\right)^2} \leq \tau \tag{21}$$

At this stage, by some elementary calculations, if we set

$$\tau = \frac{1}{2} \text{ and } \theta = \frac{1}{3\sqrt{n}}, \tag{22}$$

an upper bound for the left side of the inequality (21) is $0.4487 \leq 0.5$, which means that the inequality (21) certainly holds. In this case, we concludes that $(x, s) > 0$ and $\Psi_s(v) \leq \frac{1}{2}$ are maintained during the algorithm. Thus the algorithm is well defined.

G. Iteration complexity

In the previous subsections we have found that if at the start of an iteration the iterates satisfy $\Psi_s(v) \leq \tau$, then after a full step and an μ-update, the iterates satisfy $\Psi_s\left(x^+, s^+; \mu^+\right) \leq \tau$, where τ and θ as defined in (22).

Lemma 10: (Lemma II.17 in [2]) Assume that x^0 and s^0 are strictly feasible, and let $\mu^0 = \frac{\left(x^0\right)^T s^0}{n}$. Moreover, let x^k and s^k be the vectors obtained after k iterations. If the barrier parameter μ is repeatedly multiplied by $1-\theta$, with $0 < \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta} \log \frac{\mu^0 n}{\varepsilon} \right\rceil$$

iterations we have $x^T s \leq \varepsilon$.

For $\theta = \frac{1}{3\sqrt{n}}$, the following theorem holds trivially.

Theorem 11: Setting $\tau = \frac{1}{2}$ and $\theta = \frac{1}{3\sqrt{n}}$, the initial duality gap is $\left(x^0\right)^T s^0 = n\mu^0$, the full modified-Newton step primal-dual IPMs for LO has the iteration bound

$$\mathcal{O}\left(3\sqrt{n} \log \frac{\mu^0 n}{\varepsilon}\right).$$

V. Conclusion

The simple function used in this paper has the finite barrier property, the complexity result for the IPM based on this function admits the best-known iteration bound for LO.

It should be noted that the direction used in this paper can also be obtained by an equivalent algebraic transformation on the central path, where the Taylor series is used to derive the new direction, the interested reader may refer [14] for further understanding.

Our further research line may focus on two aspects. One is to find a better analysis method, by which the simple function can be used in the analysis of large-update methods. The other is doing the numerical tests for a few improvements of algorithms and exploiting its performance for real-world problems.

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