On a Generalization of Cubic Spline Interpolation

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***Abstract*—Based on analysis of basic cubic spline interpolation, the clamped cubic spline interpolation is generalized in this paper. The methods are presented on the condition that the first derivative and second derivative of arbitrary node are given. The Clamped spline and Curvature-adjusted cubic spline are also generalized. The methods are presented on the condition that the first derivatives of arbitrary two nodes or second derivatives of arbitrary two node are given. At last, these calculation methods are illustrated through examples.**

***Index Terms*—cubic spline function, boundary, clamped spline, first derivative, second derivative**

I. Introduction

Splines and particularly cubic splines are very popular models for interpolation. Historically, a ``spline'' was a common drafting tool, a flexible rod, that was used to help draw smooth curves connecting widely spaced points. The cubic spline curve accomplishes the same result for an interpolation problem. The spline technology has applications in CAD, CAM, and computer graphics systems. We describe cubic splines in this note and discuss their use in interpolation and curve fitting.. The cubic spline interpolation is a piecewise continuous curve, passing through each of the values in the table. There is a separate cubic polynomial for each interval, each with its own coefficients. For the approximation of gradients from data values at vertices of a uniform grid, P. Sablonnière[1] compare two methods based on cubic spline interpolation with a classical method based on finite differences. For univariate cubic splines, p. Sablonnière use the so-called de Boor's Not a Knot property and a new method giving pretty good slopes. J. S. Behar, S. J. Estrada and M. V. Hernández [2] have developed a G 2-continuous cubic A-spline scheme smoothing the polygon defined by the line segments joining consecutive data points, such that the spline curve lies completely on the same side of the boundary polygon as the data. The proposed A-spline scheme provides an efficient method for generating a smooth robot's path that avoids corners or polygonal objects for a given planned path, for designing a smooth curve on a polygonal piece of material, etc. Petrinovic, Davor [3] presents two formulations of causal cubic splines with equidistant knots. Both are based on a causal direct B-spline filter with parallel or cascade implementation. In either implementation, the causal part of the impulse response is realized with an efficient infinite-impulse-response (IIR) structure, while only the anticausal part is approximated with a finite-order finite-impulse-response (FIR) filter.

The traditional cubic spline interpolation is generalized in this paper. The methods are presented on the condition that the first derivative and second derivative of arbitrary node are given. The Clamped spline and Curvature-adjusted cubic spline are also generalized.

II. Cubic spline interpolation

Suppose that $\{(x_i, y_i)\}_{i=0}^{N}$ are $N+1$ points, where $a = x_0, x_1, \cdots, x_n = b$. The function $S(x)$ is called a cubic spline if there exit N cubic polynomials and satisfy the properties:

I. $S(x)$ is a cubic polynomial on $[x_{i-1}, x_i]$ ($i = 1,2,\cdots,n$)

II. $S(x_i^-) = S(x_i^+)$ $(i = 1,2,\cdots,n-1)$

III. $S'(x_i^-) = S'(x_i^+)$ $(i = 1,2,\cdots,n-1)$

IV. $S''(x_i^-) = S''(x_i^+)$ $(i = 1,2,\cdots,n-1)$

V. $S(x_i) = y_i (i = 0,1,\cdots,n)$.

Each cubic polynomial has four unknown constants, hence there are $4N$ coefficients to be determined. The data points supply $N+1$ conditions, and properties II, III and IV each supply $N-1$ conditions. Hence, $N+1+3(N-1) = 4N-2$ conditions are specified. This leaves us two additional degrees of freedom. The choice of these two extra conditions determines the type of the cubic spline obtained.

One of the following sets of boundary conditions is satisfied [4][5][6][7]:

(i) Clamped spline: $S'(x_0) = y_0^{'}$, $S'(x_n) = y_n^{'}$

doi:10.4304/jsw.6.9.1632-1639

(ii) Curvature-adjusted cubic spline: $S''(x_0) = y_0''$, $S''(x_n) = y_n''$

(iii) Periodic spline: $S(x_0) = S(x_n)$, $S'(x_0) = S'(x_n)$, $S''(x_0) = S''(x_n)$.

IV. GENERALIZATION I

A. New conditions of the spline interpolation

The new boundary conditions are presented on the condition that the first derivative and second derivative of arbitrary node are given. Three instances are taken into account:

(i) First condition: $S'(x_0) = y_0'$, $S''(x_0) = y_0''$

(ii) Second condition: $S'(x_n) = y_n'$, $S''(x_n) = y_n''$

(iii) Third condition: $S'(x_j) = y_j'$, $S''(x_j) = y_j''$

Three situations are analyzed and calculated as below respectively.

(i) First condition: $S'(x_0) = y_0'$ and $S''(x_0) = y_0''$ are known.

With $S(x_0) = y_0$, $S(x_1) = y_1$, $S'(x_0) = y_0'$ and $S''(x_0) = y_0''$,the spline is obtained by method of undetermined coefficient on interval $[x_0, x_1]$.

$$S(x) = y_0[1-(\frac{x-x_0}{h_0})^3] + y_1(\frac{x-x_0}{h_0})^3 + h_0 y_0'[-(\frac{x-x_0}{h_0})^3 + \frac{x-x_0}{h_0}] + \frac{h_0^2}{2} y_0''[-(\frac{x-x_0}{h_0})^3 + (\frac{x-x_0}{h_0})^2] \tag{1}$$

where $h_0 = x_1 - x_0$.Differentiate the numerators twice and get

$$S'(x_1) = \frac{3(y_1 - y_0)}{h_0} - 2y_0' - \frac{h_0 y_0''}{2} \tag{2}$$

$$S''(x_1) = \frac{6(y_1 - y_0)}{h_0^2} - \frac{6y_0'}{h_0} - 2y_0'' \tag{3}$$

In the same way, with $S(x_1) = y_1$, $S(x_2) = y_2$, $S'(x_1)$ and $S''(x_1)$,the cubic spline is also obtained by on interval $[x_1, x_2]$. $S'(x_2)$ and $S''(x_2)$ can get, which are the boundary condition of cubic spline on interval $[x_2, x_3]$.

Using the same methods, the cubic spline is also obtained by on interval $[x_i, x_{i+1}]$

$$S(x) = y_i[1-(\frac{x-x_i}{h_i})^3] + y_{i+1}(\frac{x-x_i}{h_i})^3 + h_i y_i'[-(\frac{x-x_i}{h_i})^3 + \frac{x-x_i}{h_i}] + \frac{h_i^2}{2} y_i''[-(\frac{x-x_i}{h_i})^3 + (\frac{x-x_i}{h_i})^2] \tag{4}$$

where $h_i = x_{i+1} - x_i$. The first derivative and second derivative at x_{i+1} are

$$S'(x_{i+1}) = \frac{3(y_{i+1} - y_i)}{h_i} - 2y_i' - \frac{h_i y_i''}{2} \tag{5}$$

$$S''(x_{i+1}) = \frac{6(y_{i+1} - y_i)}{h_i^2} - \frac{6y_i'}{h_i} - 2y_i'' \tag{6}$$

Which are the boundary condition of spline on next interval.

(ii) Second condition: $S'(x_n) = y_n'$, $S''(x_n) = y_n''$ are known.

Similar to the first condition, with $S(x_{n-1}) = y_{n-1}$, $S(x_n) = y_n$, $S'(x_n) = y_n'$ and $S''(x_n) = y_n''$, the cubic spline is also obtained by on interval $[x_{n-1}, x_n]$. $S'(x_{n-1})$ and $S''(x_{n-1})$ are obtained thereby, which are the boundary condition of cubic spline on interval $[x_{n-2}, x_{n-1}]$. For the same method, the cubic spline on interval $[x_i, x_{i+1}]$ is

$$S(x) = y_{i+1}[1+(\frac{x-x_{i+1}}{h_i})^3] - y_i(\frac{x-x_{i+1}}{h_i})^3 - h_i y_{i+1}'[(\frac{x-x_{i+1}}{h_i})^3 - \frac{x-x_{i+1}}{h_i}] + \frac{h_i^2}{2} y_{i+1}''[(\frac{x-x_{i+1}}{h_i})^3 + (\frac{x-x_{i+1}}{h_i})^2] \tag{7}$$

where $h_i = x_{i+1} - x_i$. The first derivative and second derivative at x_i are

$$S'(x_i) = \frac{3(y_{i+1} - y_i)}{h_i} - 2y_{i+1}' + \frac{h_i y_{i+1}''}{2} \tag{8}$$

$$S''(x_i) = -\frac{6(y_{i+1} - y_i)}{h_i^2} + \frac{6y_{i+1}'}{h_i} - 2y_{i+1}'' \tag{9}$$

(iii) Third condition: $S'(x_j) = y_j'$, $S''(x_j) = y_j''$ are known.

The third condition can use the results of first condition and second condition directly.

①Using equation (5) and (6), the first derivative and second derivative of $x_{j+1}, x_{j+2}, \cdots, x_{n-1}$ can be obtained in order. Using equation (4), the cubic spline is also obtained by on interval $[x_j, x_n]$.

② Using equation (8) and (9), the first derivative and second derivative of $x_{j-1}, x_{j-2}, \cdots, x_1$ can be obtained in order.. Using equation (7), the cubic spline is also obtained by on interval $[x_0, x_j]$.

B. Examples

Example 1: Find the cubic spline for the points (-1,14), (0,-3), (1,8), (4,32), and (5,16), where $S'(-1) = -31$ and $S''(-1) = 24$.

Using equation (5) and (6), it is easy to find the solution $S'(0) = -1$, $S''(0) = 36$, $S'(1) = 17$, $S''(1) = 0$, $S'(4) = -10$ and $S''(4) = -18$.

Using equation (4), the solution is

$$S(x) = \begin{cases} 2x^3 + 18x^2 - x - 3 & -1 \le x \le 0 \\ -6x^3 + 18x^2 - x - 3 & 0 < x \le 1 \\ -x^3 + 3x^2 + 14x - 8 & 1 < x \le 4 \\ 3x^3 - 45x^2 + 206x - 264 & 4 < x \le 5 \end{cases}$$

The cubic spline is shown in figure 1.

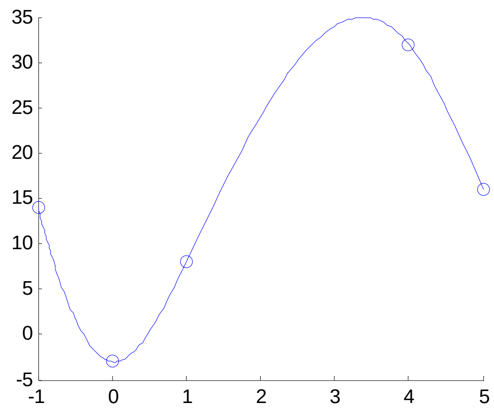

Fig. 1: The cubic spline on the first condition

Example 2: Find the cubic spline for the points (0,0), (1,1), (2,9), (3,34), and (4,84), where $S'(4) = 66$ and $S''(4) = 36$.

Using equation (8) and (9), it is easy to find the solution $S'(3) = 36$, $S''(3) = 24$, $S'(2) = 15$, $S''(2) = 18$, $S'(1) = 3$ and $S''(1) = 6$.

Using equation (7), the solution is

$$S(x) = \begin{cases} x^3 & 0 \le x \le 1 \\ 2x^3 - 3x^2 + 3x - 1 & 1 < x \le 2 \\ x^3 + 3x^2 - 9x + 7 & 2 < x \le 3 \\ 2x^3 - 6x^2 + 18x - 20 & 3 < x \le 4 \end{cases}$$

The cubic spline is shown in figure 2.

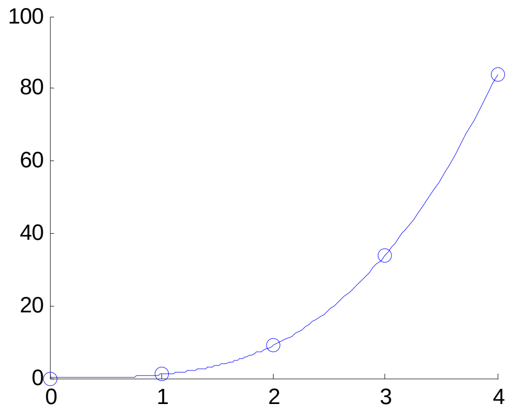

Fig. 2: The cubic spline on the second condition

Example 3: Find the cubic spline for the points (0,1), (1,0), (2,0), (3,0), and (4,0), where $S'(2) = -2$ and $S''(2) = 6$.

Using equation (5) and (6), we can find the solution $S'(3) = 1$ and $S''(3) = 0$. Using equation (4), the cubic spline of interval [2,4] can be obtained.

Using equation (8) and (9), we can find the solution $S'(1) = 7$ and $S''(1) = -24$. Using equation (7), the cubic spline of interval [0,2] can be obtained. the cubic spline of interval [0,4] is

$$S(x) = \begin{cases} -19x^3 + 45x^2 - 26x & 0 \le x \le 1 \\ 5x^3 - 27x^2 + 46x - 24 & 1 < x \le 2 \\ -x^3 + 9x^2 - 26x + 24 & 2 < x \le 3 \\ -x^3 + 9x^2 - 26x + 24 & 3 < x \le 4 \end{cases}$$

The cubic spline is shown in figure 3.

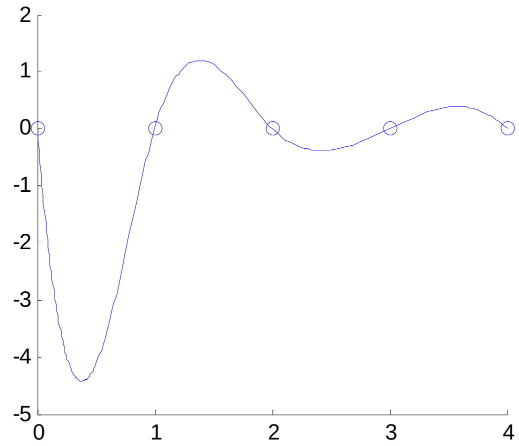

Fig. 3: The cubic spline on the third condition

III. Generalization II

A. New conditions of the clamped spline

For the clamped boundary conditions, we use $m_i = S'(x_i)(i = 0,1,\cdots,n)$. Specify $m_0 = y_0'$, $m_n = y_n'$, we obtain $N-1$ linear equations involving the coefficients $m_1, m_2, \cdots, m_{n-1}$.

$$\begin{bmatrix} 2 & \alpha_1 & & & & \\ 1-\alpha_2 & 2 & \alpha_2 & & & \\ & 1-\alpha_3 & 2 & \alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-\alpha_{n-2} & 2 & \alpha_{n-2} \\ & & & & 1-\alpha_{n-1} & 2 \end{bmatrix} \times \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ \\ m_{n-1} \end{bmatrix} = \begin{bmatrix} \beta_1 - (1-\alpha_1)m_0 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} - \alpha_{n-1}m_n \end{bmatrix} \tag{10}$$

where $h_{i-1} = x_i - x_{i-1}$, $\alpha_i = \dfrac{h_{i-1}}{h_{i-1} + h_i}$,

$\beta_i = 3[(1-\alpha_i)\dfrac{y_i - y_{i-1}}{h_{i-1}} + \alpha_i \dfrac{y_{i+1} - y_i}{h_i}]$ $(i = 1,2,\cdots,n-1)$.

This is a tridiagonal linear system. The $m_1, m_2, \cdots, m_{n-1}$ are obtain by Crout Factorization algorithm. The result is the following expression for the cubic function $S_i(x)$ on $[x_{i-1}, x_i](i = 1,2,\cdots,n)$.

$$S_i(x) = \varphi_0(\frac{x - x_{i-1}}{h_{i-1}})y_{i-1} + \varphi_1(\frac{x - x_{i-1}}{h_{i-1}})y_i + h_{i-1}\psi_0(\frac{x - x_{i-1}}{h_{i-1}})m_{i-1} + h_{i-1}\psi_1(\frac{x - x_{i-1}}{h_{i-1}})m_i \tag{11}$$

where $\varphi_0(x) = (x-1)^2(2x+1)$, $\varphi_1(x) = x^2(-2x+3)$, $\psi_0(x) = x(x-1)^2$ and $\psi_1(x) = x^2(x-1)$.

Consider clamped cubic spline, the first derivatives of arbitrary two nodes are given. Specify m_j and m_k ($j < k$) are known, and the value m_j and m_k are substituted into the equation (1), then the equation (j+1)~ equation (k-1) is

$$\begin{bmatrix} 2 & \alpha_{j+1} & & & \\ 1-\alpha_{j+2} & 2 & \alpha_{j+2} & & \\ & 1-\alpha_{j+3} & 2 & \alpha_{j+3} & \\ & & \ddots & \ddots & \ddots \\ & & & 1-\alpha_{k-1} & 2 \end{bmatrix} \times \begin{bmatrix} m_{j+1} \\ m_{j+2} \\ m_{j+3} \\ \vdots \\ \\ m_{k-1} \end{bmatrix} = \begin{bmatrix} \beta_{j+1} - (1-\alpha_{j+1})m_j \\ \beta_{j+2} \\ \beta_3 \\ \vdots \\ \\ \beta_{k-1} - \alpha_{k-1}m_k \end{bmatrix} \tag{12}$$

This is also a tridiagonal linear system. It is easy to find the solution $m_{j+1}, m_{j+2}, \cdots, m_{k-1}$ by Crout Factorization algorithm[1].

The value m_{j+1} is substituted into the equation (j), then the equation (1)~equation (j) is

$$\begin{bmatrix} 1-\alpha_1 & 2 & \alpha_1 & & & \\ & 1-\alpha_2 & 2 & \alpha_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-\alpha_{j-2} & 2 & \alpha_{j-2} \\ & & & & 1-\alpha_{j-1} & 2 \\ & & & & & 1-\alpha_j \end{bmatrix} \times \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ \\ m_{j-3} \\ m_{j-2} \\ m_{j-1} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \\ \vdots \\ \beta_{j-2} \\ \beta_{j-1} - \alpha_{j-1}m_j \\ \beta_j - 2m_j - \alpha_j m_{j+1} \end{bmatrix} \tag{13}$$

This is an upper-tridiagonal $j \times j$ linear system. It is easy to find the solution m_{j-1}, $m_{j-2}, m_{j-3}, \cdots, m_0$.

The value m_{k-1} is substituted into the equation (k), then the equation (k)~equation (n-1) is

$$\begin{bmatrix} \alpha_k & & & & \\ 2 & \alpha_{k+1} & & & \\ 1-\alpha_{k+2} & 2 & \alpha_{k+2} & & \\ & \ddots & \ddots & \ddots & \\ & & 1-\alpha_{n-1} & 2 & \alpha_{n-1} \end{bmatrix} \times \begin{bmatrix} m_{k+1} \\ m_{k+2} \\ m_{k+3} \\ \vdots \\ \\ m_n \end{bmatrix} = \begin{bmatrix} \beta_k - 2m_k - (1-\alpha_k)m_{k-1} \\ \beta_{k+1} - (1-\alpha_{k+1})m_k \\ \beta_{k+2} \\ \vdots \\ \\ \beta_{n-1} \end{bmatrix} \tag{14}$$

This is a lower-tridiagonal $(n-k)\times(n-k)$ linear system. It is easy to find the solution m_{k+1}, m_{k+2}, $m_{k+3}, \cdots, m_n$. Using equation (11), the cubic spline can be obtained.

B. Example of New conditions of the clamped spline

Example 4: Find the cubic spline for the points (0,2), (1,2), (2,0.5), (3,3), (4,-2), (5,1), (6,9), (7,2), (8,1.5), (9,2), (10,3) and (11,2), where $m_3 = -1$ and $m_8 = 0.5$.

The values $m_3 = -1$ and $m_8 = 0.5$ are substituted into equations (1), then obtain the equations

$$\begin{bmatrix} 0.5 & 2 & 0.5 & & & & & & & \\ & 0.5 & 2 & & & & & & & \\ & & 0.5 & 0.5 & & & & & & \\ & & & 2 & 0.5 & & & & & \\ & & & 0.5 & 2 & 0.5 & & & & \\ & & & & 0.5 & 2 & 0.5 & & & \\ & & & & & 0.5 & 2 & & & \\ & & & & & & 0.5 & 0.5 & & \\ & & & & & & & 2 & 0.5 & \\ & & & & & & & 0.5 & 2 & 0.5 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_9 \\ m_{10} \\ m_{11} \end{bmatrix} = \begin{bmatrix} -2.25 \\ 2 \\ -1.75 \\ -2.5 \\ 16.5 \\ 1.5 \\ -11.5 \\ -2.5 \\ 2 \\ 1.5 \end{bmatrix}$$

The equation (4)~equation (7) is

$$\begin{bmatrix} 2 & 0.5 & & \\ 0.5 & 2 & 0.5 & \\ & 0.5 & 2 & 0.5 \\ & & 0.5 & 2 \end{bmatrix} \begin{bmatrix} m_4 \\ m_5 \\ m_6 \\ m_7 \end{bmatrix} = \begin{bmatrix} -2.5 \\ 16.5 \\ 1.5 \\ -11.5 \end{bmatrix}$$

This is also a tridiagonal linear system. It is easy to find the solution $m_4 = -3.5407$, $m_5 = 9.1627$, $m_6 = -0.1100$ and $m_7 = -5.7225$.

The value $m_4 = -3.5407$ is substituted onto equation (3) and the equation (1)~equation (3) is

$$\begin{bmatrix} 0.5 & 2 & 0.5 \\ & 0.5 & 2 \\ & & 0.5 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -2.25 \\ 2 \\ 0.0204 \end{bmatrix}$$

This is an upper-tridiagonal linear system. It is easy to find the solution $m_2 = 0.0408$, $m_1 = 3.8368$ and $m_0 = -19.8880$ by turns.

The value $m_7 = -5.7225$ is substituted onto equation (8) and the equation (8)~equation (10) is

$$\begin{bmatrix} 0.5 & & \\ 2 & 0.5 & \\ 0.5 & 2 & 0.5 \end{bmatrix} \begin{bmatrix} m_9 \\ m_{10} \\ m_{11} \end{bmatrix} = \begin{bmatrix} 0.3613 \\ 2 \\ 1.5 \end{bmatrix}$$

This is a lower-tridiagonal linear system. It is easy to find the solution $m_9 = 0.7225$, $m_{10} = 1.1100$ and $m_{11} = -2.16125$ by turns.

Using equation (11), the cubic spline of interval [0,11] can be obtained.

If $0 \le x < 1$,

$$S(x) = -16.0526x^3 + 35.9426x^2 + 19.89x + 2$$

If $1 \le x < 2$,

$$S(x) = 6.878x^3 - 32.8493x^2 + 48.9019x - 20.9306$$

If $2 \le x < 3$,

$$S(x) = 5.4593x^3 - 58.5933x^2 + 203.1579x + 226.5359$$

If $3 \le x < 4$

$$S(x) = -0.3780x^3 + 11.4545x^2 - 77.0335x + 147.0526$$

If $4 \le x < 5$

$$S(x) = -6.9474x^3 + 109.9952x^2 - 569.7368x + 968.2249$$

If $5 \le x < 6$

$$S(x) = 8.1675x^3 - 162.0718x^2 + 1062.7x - 2296.6$$

If $7 \le x < 8$

$$S(x) = -4.2225x^3 + 98.1172x^2 - 758.6579x + 1953.2$$

If $8 \le x < 9$

$$S(x) = 2.2225x^3 - 56.5622x^2 + 478.7775x - 1346.7$$

If $9 \le x < 10$

$$S(x) = -2.1675x^3 + 61.9665x^2 - 587.9809x + 1853.6$$

If $10 \le x < 11$

$$S(x) = 0.9474x^3 - 31.4785x^2 + 285.3206x - 955.4689$$

The cubic spline is shown in Figure 4.

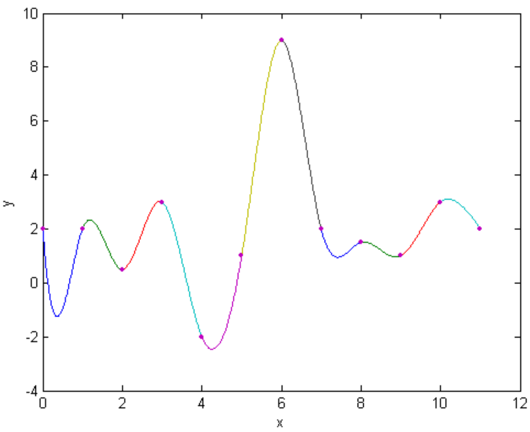

Fig. 4 The cubic spline of example 4

C. New conditions of Curvature-adjusted cubic spline

For the Curvature-adjusted cubic spline conditions, we use $m_i = S'(x_i)(i = 0,1,\cdots,n)$. Specify $S''(x_0) = y_0''$ and $S''(x_n) = y_n''$ are known, we obtain $N-1$ linear equations involving the coefficients $m_0, m_1, \cdots, m_n$.

$$\begin{bmatrix} 2 & 1 & & & & \\ 1-\alpha_1 & 2 & \alpha_1 & & & \\ & 1-\alpha_2 & 2 & \alpha_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-\alpha_{n-1} & 2 & \alpha_{n-1} \\ & & & & 1 & 2 \end{bmatrix} \times \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \\ \\ m_n \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \quad (15)$$

where $h_{i-1} = x_i - x_{i-1}$, $\alpha_i = \dfrac{h_{i-1}}{h_{i-1}+h_i}$,

$$\beta_i = 3[(1-\alpha_i)\frac{y_i - y_{i-1}}{h_{i-1}} + \alpha_i \frac{y_{i+1} - y_i}{h_i}](i = 1,2,\cdots,n-1).$$

$$\beta_0 = 3\frac{y_1 - y_0}{h_1} - \frac{h_1}{2}y_0'' , \quad \beta_n = 3\frac{y_n - y_{n-1}}{h_n} + \frac{h_n}{2}y_n''.$$

Using equation (11), the cubic spline is also obtained.

Consider Curvature-adjusted spline, the seond derivatives of arbitrary two nodes are given. Specify $S''(x_j) = y_j''$ and $S''(x_k) = y_k''$ ($j < k$) are known, we obtain $k - j + 1$ linear equations involving the coefficients $m_j, m_{j+1}, \cdots, m_k$.

$$\begin{bmatrix} 2 & 1 & & & & \\ 1-\alpha_{j+1} & 2 & \alpha_{j+1} & & & \\ & 1-\alpha_{j+2} & 2 & \alpha_{j+2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-\alpha_{k-1} & 2 & \alpha_{k-1} \\ & & & & 1 & 2 \end{bmatrix} \times \begin{bmatrix} m_j \\ m_{j+1} \\ m_{j+2} \\ \vdots \\ \\ m_k \end{bmatrix} = \begin{bmatrix} \beta_j \\ \beta_{j+1} \\ \beta_{j+2} \\ \vdots \\ \beta_{k-1} \\ \beta_k \end{bmatrix} \quad (16)$$

where $h_{i-1} = x_i - x_{i-1}$, $\alpha_i = \dfrac{h_{i-1}}{h_{i-1}+h_i}$,

$$\beta_i = 3[(1-\alpha_i)\frac{y_i - y_{i-1}}{h_{i-1}} + \alpha_i \frac{y_{i+1} - y_i}{h_i}].$$

$$(i = j+1, j+2, \cdots, k-1)$$

$$\beta_j = 3\frac{y_{j+1} - y_j}{h_{j+1}} - \frac{h_1}{2}y_j'' , \quad \beta_k = 3\frac{y_k - y_{k-1}}{h_k} + \frac{h_k}{2}y_k''.$$

Using equation (11), the cubic spline is also obtained by on interval $[x_j, x_k]$.

①Using equation (5) and (6), the first derivative and second derivative of $x_{k+1}, x_{k+2}, \cdots, x_{n-1}$ can be obtained in order. Using equation (4), the cubic spline is also obtained by on interval $[x_k, x_n]$.

② Using equation (8) and (9), the first derivative and second derivative of $x_{j-1}, x_{j-2}, \cdots, x_1$ can be obtained in order.. Using equation (7), the cubic spline is also obtained by on interval $[x_0, x_j]$.

D. Examples of New conditions of Curvature-adjusted spline

Example 5: Find the cubic spline for the points (-1,90), (0,0), (1,0), (2,0), (3,0), and (4,0), where $S''(0) = 90$ and $S''(3) = 0$.

Using equation (16), then we obtain the equations

$$\begin{bmatrix} 2 & 1 & & \\ 0.5 & 2 & 0.5 & \\ & 0.5 & 2 & 1 \\ & & 1 & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} -45 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $m_1 = -26$, $m_2 = 7$, $m_3 = -2$ and $m_4 = 1$.

Using equation (11), the cubic spline of interval [0,3] is

$$S(x) = \begin{cases} -19x^3 + 45x^2 - 26x & 0 \le x \le 1 \\ 5x^3 - 27x^2 + 46x - 24 & 1 < x \le 2 \\ -x^3 + 9x^2 - 26x + 24 & 2 < x \le 3 \end{cases}$$

From the above cubic spline , we find $S''(3) = 0$ and $S''(0) = 90$.

Using equation (5) and (6), we can find the solution $S'(4) = -2$ and $S''(4) = -6$. Using equation (4), the cubic spline of interval [3,4] can be obtained.

Using equation (8) and (9), we can find the solution $S'(-1) = -173$ and $S''(-1) = 204$. Using equation (7), the cubic spline of interval [-1,0] can be obtained.

The cubic spline of interval [0,4] is

$$S(x) = \begin{cases} -19x^3 + 45x^2 - 26x & -1 \le x \le 1 \\ 5x^3 - 27x^2 + 46x - 24 & 1 < x \le 2 \\ -x^3 + 9x^2 - 26x + 24 & 2 < x \le 4 \end{cases}$$

The cubic spline is shown in figure 5.

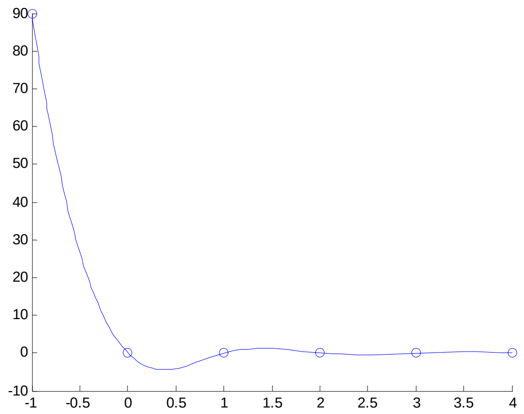

Fig. 5 The cubic spline of example 5

V. CONCLUSIONS

Cubic splines are popular because they are easy to implement and produce a curve that appears to be seamless. As we have seen, a straight polynomial interpolation of evenly spaced data tends to build in distortions near the edges of the table. Cubic splines avoid this problem, but they are only piecewise continuous, meaning that a sufficiently high derivative (third) is discontinuous. So if the application is sensitive to the smoothness of derivatives higher than second, cubic splines may not be the best choice. The traditional cubic spline interpolation is generalized. In this paper the Crout Factorization algorithm is not discussed in detail.

Other conditions of the clamped spline are researched in this paper, and other spline can be also deal with similarly. Generalization I is less of computation than the traditional cubic spline interpolation, because it needn't solve equation systems. Generalization II are supplement of the Clamped spline and Curvature-adjusted cubic spline.

The strategies for the end-point constraints are summarized in table 1.

TABLE I.
END-POINT CONSTRAINTS OF CUBIC SPLINE

Description of the strategy	method
First condition: $S'(x_0)=y_0'$, $S''(x_0)=y_0''$	equation (5), equation (6) and equation (4)
Second condition: $S'(x_n)=y_n'$, $S''(x_n)=y_n''$	equation (8), equation (9) and equation (7)
Third condition: $S'(x_j)=y_j'$, $S''(x_j)=y_j''$	interval $[x_j, x_n]$: equation (5), equation (6) and equation (4) interval $[x_0, x_j]$: equation (8), equation (9) and equation (7)
Clamped spline: $S'(x_0)=y_0'$, $S'(x_n)=y_n'$	equation (10) and equation (11)
m_j and m_k ($j<k$) are known	equation (12), equation (13) and equation (14)
Curvature-adjusted cubic spline: $S''(x_0)=y_0''$, $S''(x_n)=y_n''$	equation (15) and equation (11)
$S''(x_j)=y_j''$ and $S''(x_k)=y_k''$ ($j<k$) are known	interval $[x_j, x_k]$: equation (16), equation (11) interval $[x_k, x_n]$: equation (16), equation (5), equation (6) and equation (4) interval $[x_0, x_j]$: equation (16), equation (8), equation (9) and equation (7)

ACKNOWLEDGMENT

This work was partially supported by National Basic Research Program of Jiangsu Province University (08KJB520003) and the National Natural Science Foundation of China under Grant No.60773059.

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