# Multiplication Operation on Fuzzy Numbers

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***Abstract*—A fuzzy number is simply an ordinary number whose precise value is somewhat uncertain. Fuzzy numbers are used in statistics, computer programming, engineering, and experimental science. The arithmetic operators on fuzzy numbers are basic content in fuzzy mathematics. Operation of fuzzy number can be generalized from that of crisp interval. The operations of interval are discussed. Multiplication operation on fuzzy numbers is defined by the extension principle. Based on extension principle, nonlinear programming method, analytical method, computer drawing method and computer simulation method are used for solving multiplication operation of two fuzzy numbers. The nonlinear programming method is a precise method also, but it only gets a membership value as given number and it is a difficult problem for solving nonlinear programming. The analytical method is most precise, but it is hard to $\alpha$ -cuts interval when the membership function is complicated. The computer drawing method is simple, but it need calculate the $\alpha$ -cuts interval. The computer simulation method is the most simple, and it has wide applicability, but the membership function is rough. Each method is illuminated by examples.**

***Index Terms*—fuzzy number, membership function, extension principle, $\alpha$ -cuts; nonlinear programming**

## I. INTRODUCTION

In most of cases in our life, the data obtained for decision making are only approximately known. In1965, Zadeh [1] introduced the concept of fuzzy set theory to meet those problems. In 1978, Dubois and Prade defined any of the fuzzy numbers as a fuzzy subset of the real line [2]. Fuzzy numbers allow us to make the mathematical model of linguistic variable or fuzzy environment. A fuzzy number is a quantity whose value is imprecise, rather than exact as is the case with "ordinary" (single-valued) numbers [3-6]. Any fuzzy number can be thought of as a function whose domain is a specified set. In many respects, fuzzy numbers depict the physical world more realistically than single-valued numbers. Fuzzy numbers are used in statistics, computer programming, engineering (especially communications), and experimental science. S. M. Wang and J. Watada [6] discuss the laws of large numbers for T-independent L-R fuzzy variables based on continuous archimedean t-norm and expected value of fuzzy variable. G. X. Wang, Q. Zhang and X. J. Cui [8] define a special sort discrete fuzzy numbers - discrete fuzzy number on a fixed set with finite support set, and then obtain a representation theorem of such discrete fuzzy numbers, study the operations of scalar product, addition and multiplication, and obtain some results. H. M. Lee and L. Lin [9] weighted triangular fuzzy numbers to tackle the rate of aggregative risk in fuzzy circumstances during any phase of the software development life cycle. X. W. Zhou, L. P. Wang and B. H. Zheng [10] calculated concentration of pollution by means of triangle fuzzy number and established fuzzy risk assessment model of the potential ecological risk index. D. Sanchez, M. Delgado and M. A. Vila [11] define imprecise quantities on the basis of a new representation of imprecision introduced by the authors called RL-representation and show that the imprecision of the quantities being operated can be increased, preserved or diminished.

The concept takes into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty. The arithmetic operators on fuzzy numbers are basic content in fuzzy mathematics. Multiplication operation on fuzzy numbers is defined by the extension principle. The procedure of addition or subtraction is simple, but the procedure of multiplication or division is complex. The nonlinear programming, analytical method, computer drawing and computer simulation method are used for solving multiplication operation of two fuzzy numbers. The procedure of division is similar.

## II. CONCEPT OF FUZZY NUMBER

### A. Fuzzy Number

If a fuzzy set is convex and normalized, and its membership function is defined in ***R*** and piecewise continuous, it is called as fuzzy number. So fuzzy number

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(fuzzy set) represents a real number interval whose boundary is fuzzy.

Fuzzy number is expressed as a fuzzy set defining a fuzzy interval in the real number ***R***. Since the boundary of this interval is ambiguous, the interval is also a fuzzy set. Generally a fuzzy interval is represented by two end points $a_1$ and $a_3$ and a peak point $a_2$ as $[a_1, a_2, a_3]$ (Figure 1). The $\alpha$ -cut operation can be also applied to the fuzzy number. If we denote $\alpha$ -cut interval for fuzzy number $A$ as $A_\alpha$ , the obtained interval $A_\alpha$ is defined as

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$

We can also know that it is an ordinary crisp interval (Figure 2).

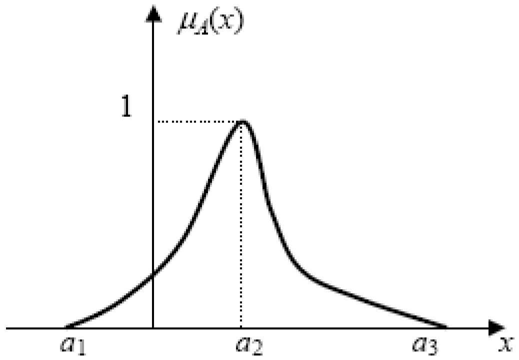


Figure 1 Fuzzy Number $A = [a_1, a_2, a_3]$

Fuzzy number should be normalized and convex. Here the condition of normalization implies that maximum membership value is 1.

$$\exists x_0 \in R, \mu_{\tilde{A}}(x_0) = 1$$

*B. Operation of $\alpha$ -cut Interval*

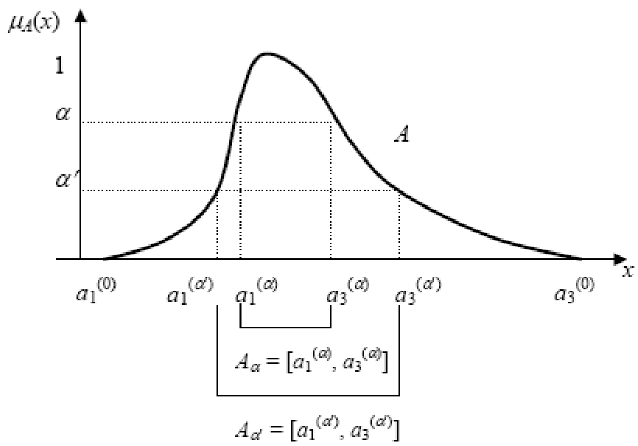


Figure 2 $\alpha$ -cut of fuzzy number

$$(\alpha' < \alpha) \Rightarrow (A_\alpha \subset A_{\alpha'})$$

The convex condition is that the line by $\alpha$ -cut is continuous and $\alpha$ -cut interval satisfies the following relation.

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$

$$(\alpha' < \alpha) \Rightarrow (a_1^{(\alpha')} \leq a_1^{(\alpha)}, a_3^{(\alpha')} \geq a_3^{(\alpha)})$$

The convex condition may also be written as,

$$(\alpha' < \alpha) \Rightarrow (A_\alpha \subset A_{\alpha'})$$

Operation of fuzzy number can be generalized from that of crisp interval. Let's have a look at the operations of interval.

$$\forall a_1, a_3, b_1, b_3 \in \mathrm{R}$$

$$A = [a_1, a_3], B = [b_1, b_3]$$

Assuming $A$ and $B$ as numbers expressed as interval, main operations of interval are

i) Addition

$$[a_1, a_3] (+) [b_1, b_3] = [a_1 + b_1, a_3 + b_3]$$

ii) Subtraction

$$[a_1, a_3] (-) [b_1, b_3] = [a_1 - b_3, a_3 - b_1]$$

iii) Multiplication

$$[a_1, a_3] (\bullet) [b_1, b_3] = [a_1 \bullet b_1 \wedge a_1 \bullet b_3 \wedge a_3 \bullet b_1 \wedge a_3 \bullet b_3,$$
$$a_1 \bullet b_1 \vee a_1 \bullet b_3 \vee a_3 \bullet b_1 \vee a_3 \bullet b_3]$$

iv) Division

$$[a_1, a_3] (/) [b_1, b_3] = [a_1 / b_1 \wedge a_1 / b_3 \wedge a_3 / b_1 \wedge a_3 / b_3,$$
$$a_1 / b_1 \vee a_1 / b_3 \vee a_3 / b_1 \vee a_3 / b_3]$$

excluding the case $b_1 = 0$ or $b_3 = 0$

v) Inverse interval

$$[a_1, a_3]^{-1} = [1 / a_1 \wedge 1 / a_3, 1 / a_1 \vee 1 / a_3]$$

excluding the case $a_1 = 0$ or $a_3 = 0$

When previous sets $A$ and $B$ is defined in the positive real number **R+**, the operations of multiplication, division, and inverse interval are written as,

iii′) Multiplication

$$[a_1, a_3] (\bullet) [b_1, b_3] = [a_1 \bullet b_1, a_3 \bullet b_3]$$

iv′) Division

$$[a_1, a_3] (/) [b_1, b_3] = [a_1 / b_3, a_3 / b_1]$$

v) Inverse Interval

$$[a_1, a_3]^{-1} = [1 / a_3, 1 / a_1]$$

vi) Minimum

$$[a_1, a_3] (\wedge) [b_1, b_3] = [a_1 \wedge b_1, a_3 \wedge b_3]$$

vii) Maximum

$$[a_1, a_3] (\vee) [b_1, b_3] = [a_1 \vee b_1, a_3 \vee b_3]$$

**Example 1** There are two intervals A and B,

$A = [3, 5]$, $B = [-2, 7]$

Then following operation might be set.

$A(+)B$=[3-2, 5+7]=[1, 12]

$A(-)B$=[3-7, 5-(-2)]=[-4, 7]

$A(\bullet)B$=[3•(-2) $\wedge$3•7$\wedge$5•(-2) $\wedge$5•7,

3•(-2) $\vee$3•7$\vee$5•(-2) $\vee$5•7]

=[-10, 35]

$A(/)B$=[3/(-2) $\wedge$3/7$\wedge$5/(-2) $\wedge$5/7,

3/(-2) $\vee$3/7$\vee$5/(-2) $\vee$5/7]

=[-2.5, 5/7]

$A^{-1}=[3,5]^{-1}$ = [1/5, 1/3]

$B^{-1}=[-2,7]^{-1}$ = [1/(-2) $\wedge$1/7, 1/(-2) $\vee$1/7]

=[-1/2, 1/7]

$A(\wedge)B$=[3$\wedge$(−2) ,5$\wedge$7]

=[-2, 5]

$A(\vee)B$=[3$\vee$(−2) ,5$\vee$7]

=[3, 7]

*C. Operation of fuzzy numbers*

Based on the extension principle, arithmetic operations on fuzzy numbers are defined by following:

If $\widetilde{M}$ and $\widetilde{N}$ are fuzzy numbers, membership of $\widetilde{M}(*)\widetilde{N}$ is defined as follow:

$$\mu_{\widetilde{M}(*)\widetilde{N}}(z) = \sup_{z=x*y} \min\{\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)\} \quad (1)$$

Where * stands for any of the four arithmetic operations.

$$\mu_{\widetilde{M}(+)\widetilde{N}}(z) = \sup_{z=x+y} \min\{\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)\}$$

$$\mu_{\widetilde{M}(-)\widetilde{N}}(z) = \sup_{z=x-y} \min\{\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)\}$$

$$\mu_{\widetilde{M}(/)\widetilde{N}}(z) = \sup_{z=x/y} \min\{\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)\}$$

Therefore multiplication operation on fuzzy numbers is expressed as

$$\mu_{\widetilde{M}(\times)\widetilde{N}}(z) = \sup_{z=x\times y} \min\{\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)\} \quad (2)$$

The procedure of addition or subtraction is simple, but the procedure of multiplication or division is complex.

## III. Nonlinear Programming Method

Based on multiplication operation on fuzzy numbers, multiplication operation problem is formulated as a nonlinear programming.

$$\begin{aligned} &\max v \\ &s.t. \quad \mu_{\widetilde{M}}(x) \ge v \\ &\qquad \mu_{\widetilde{N}}(y) \ge v \\ &\qquad xy = z_0 \end{aligned} \quad (3)$$

Given $z_0$, we can get the maximum $v_{\max}$ of nonlinear programming (1). $v_{\max}$ is membership value of $z_0$.

**Example 2** Suppose that the membership of $\widetilde{M}$ , $\widetilde{N}$ are $\mu_{\widetilde{M}}(x) = e^{-(x-2)^2}$ (Figure 3) and $\mu_{\widetilde{N}}(y) = e^{-(y+2)^2}$ (Figure 4). $\widetilde{Q} = \widetilde{M}(\times)\widetilde{N}$ is formulated as following:

$$\begin{aligned} &\max v \\ &s.t. \quad e^{-(x-2)^2} \ge v \\ &\qquad e^{-(y+2)^2} \ge v \\ &\qquad xy = z_0 \end{aligned} \quad (4)$$

When $z_0$ =-12,-11,-10,-9,-8,-7,-6,-5,-4,-3,-2,-1 respectively, we can get the $v_{\max}$ correspondingly. The membership value of $z_0$ is shown in table 1 and membership function is shown in Figure 5.

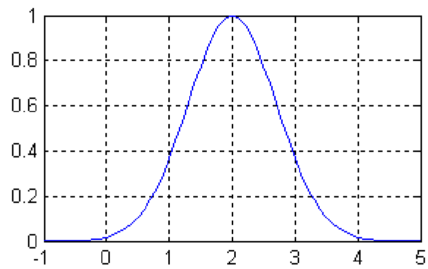


Figure 3 Membership function of $\widetilde{M}$

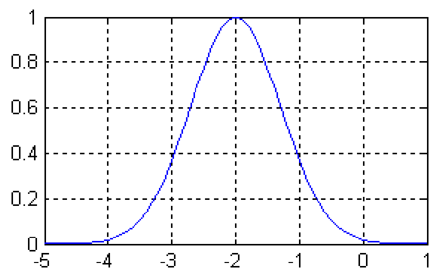


Figure 4 Membership function of $\widetilde{N}$

TABLE I.
$z_0$ AND ITS MEMBERSHIP VALUE

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_0$ | -18 | -17 | -16 | -15 | -14 | -13 | -12 | -11 | -10 | -9 |
| $v_{\max}$ | 0.0065 | 0.0110 | 0.0183 | 0.0300 | 0.0482 | 0.0759 | 0.1172 | 0.1767 | 0.2590 | 0.3679 |
| $z_0$ | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 |
| $v_{\max}$ | 0.5034 | 0.6590 | 0.8171 | 0.9458 | 1.0000 | 0.9307 | 0.7095 | 0.3679 | 0.0183 | 0.0067 |

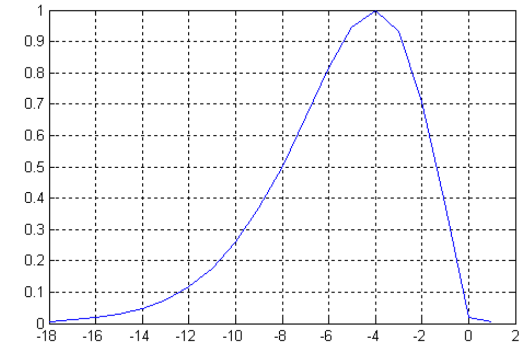


Figure 5 Membership function of $\widetilde{Q} = \widetilde{M}(\times)\widetilde{N}$

## IV. Analytical Method

### A. Decompose Theorem

It is not easy to solve nonlinear programming. We can use decompose theorem to calculate multiplication operation.

$$\begin{aligned}\mu_{\widetilde{M}(*)\widetilde{N}}(z) &= \int_{\alpha\in[0,1]} \alpha[M(*)N]_\alpha \\ &= \int_{\alpha\in[0,1]} \alpha([m_\alpha^L, m_\alpha^R](*)[n_\alpha^L, n_\alpha^R])\end{aligned} \tag{5}$$

### B. Multiplication of Triangular Fuzzy Number

Among the various shapes of fuzzy number, triangular fuzzy number (TFN) is the most popular one. It is a fuzzy number represented with three points as follows:

$A = (a, b, c)$

This representation is interpreted as membership function (Figure 6).

$$u_A(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x \le b \\ \dfrac{c-x}{c-b} & b < x \le c \\ 0 & x > c \end{cases} \tag{6}$$

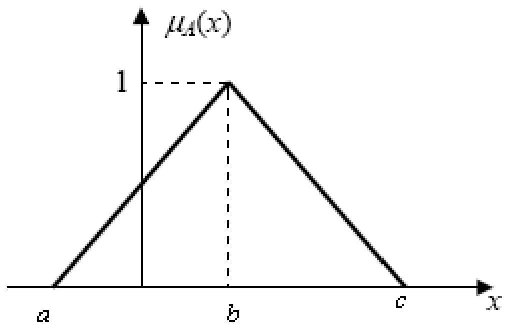


Figure 6 Triangular fuzzy number $A = (a, b, c)$

Suppose triangular fuzzy numbers $\widetilde{M}$ and $\widetilde{N}$ are defined as,

$$\widetilde{M} = (a_1, b_1, c_1),\ \widetilde{N} = (a_2, b_2, c_2)$$

Same important properties of operations on triangular fuzzy number are summarized.

(i) The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.

(ii) The results from multiplication or division are not triangular fuzzy numbers.

(iii) Max or min operation does not give triangular fuzzy number.

$$\widetilde{M}(+)\widetilde{N} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$
$$\widetilde{M}(-)\widetilde{N} = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$$
$$-(\widetilde{N}) = (-c_2, -b_2, -a_2)$$
$$M_\alpha = [m_\alpha^L, m_\alpha^R] = [(b_1 - a_1)\alpha + a_1, (b_1 - c_1)\alpha + c_1]$$
$$N_\alpha = [n_\alpha^L, n_\alpha^R] = [(b_2 - a_2)\alpha + a_2, (b_2 - c_2)\alpha + c_2]$$

$\alpha$-cuts of $\widetilde{Q} = \widetilde{M}(\times)\widetilde{N}$ is

$$[m_\alpha^L, m_\alpha^R] \times [n_\alpha^L, n_\alpha^R] = [q_\alpha^L, q_\alpha^R]$$

Where $q_\alpha^L = \min\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\}$

and $q_\alpha^R = \max\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\}$.

We suppose $c_1 \ge b_1 \ge a_1$ and $c_2 \ge b_2 \ge a_2$ firstly.

$m_\alpha^L = (b_1 - a_1)\alpha + a_1$,

$m_\alpha^R = (b_1 - c_1)\alpha + c_1,$

$n_\alpha^L = (b_2 - a_2)\alpha + a_2,$

$n_\alpha^R = (b_2 - c_2)\alpha + c_2$

$q_\alpha^L = \min\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\}$

$= m_\alpha^L n_\alpha^L$

$= (b_1 - a_1)(b_2 - a_2)\alpha^2 + (a_1 b_2 + a_2 b_1 - 2a_1 a_2)\alpha$

$+ a_1 a_2$

$q_\alpha^R = \max\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\}$

$= m_\alpha^R n_\alpha^R$

$= (b_1 - c_1)(b_2 - c_2)\alpha^2 + (c_1 b_2 + c_2 b_1 - 2c_1 c_2)\alpha$

$+ c_1 c_2$

Substituting $q_\alpha^L = z$, we get

$$\alpha = \frac{-(a_1 b_2 + a_2 b_1 - 2a_1 a_2) + \sqrt{(a_1 b_2 - a_2 b_1)^2 + 4(b_1 - c_1)(b_2 - a_2)z}}{2(b_1 - a_1)(b_2 - a_2)} \quad \text{(Omit } \alpha < 0\text{)}$$

Substituting $q_\alpha^R = z$, we get

$$\alpha = \frac{-(c_1 b_2 + c_2 b_1 - 2c_1 c_2) - \sqrt{(c_1 b_2 - c_2 b_1)^2 + 4(b_1 - c_1)(b_2 - c_2)z}}{2(b_1 - c_1)(b_2 - c_2)} \quad \text{(Omit } \alpha > 1\text{)}$$

Hence membership function of $\tilde{Q} = \tilde{M}(\times)\tilde{N}$ is

$$\mu_{\tilde{Q}}(z) = \begin{cases} \dfrac{-(a_1 b_2 + a_2 b_1 - 2a_1 a_2) + \sqrt{(a_1 b_2 - a_2 b_1)^2 + 4(b_1 - c_1)(b_2 - a_2)z}}{2(b_1 - a_1)(b_2 - a_2)} & a_1 a_2 \le z \le b_1 b_2 \\ \dfrac{-(c_1 b_2 + c_2 b_1 - 2c_1 c_2) - \sqrt{(c_1 b_2 - c_2 b_1)^2 + 4(b_1 - c_1)(b_2 - c_2)z}}{2(b_1 - c_1)(b_2 - c_2)} & b_1 b_2 \le z \le c_1 c_2 \\ 0 & otherwise \end{cases} \quad (7)$$

**Example 3** Suppose $\tilde{3} = (2,3,5)$, $\tilde{5} = (3,5,6)$, calculate membership function of $\tilde{Q} = \tilde{3} \times \tilde{5}$.

Substituting $a_1 = 2$, $b_1 = 3$, $c_1 = 5$, $a_2 = 3$, $b_2 = 5$, $c_2 = 6$ into equation (7), the result is the following expression

$$\mu_{\tilde{Q}}(z) = \begin{cases} \dfrac{1}{4}(-7 + \sqrt{1 + 8z}) & 6 \le z \le 15 \\ \dfrac{1}{4}(17 - \sqrt{49 + 8z}) & 15 < z \le 30 \\ 0 & otherwise \end{cases} \quad (8)$$

The membership functions of $\tilde{3} = (2,3,5)$ and $\tilde{5} = (3,5,6)$ are shown in Figure 7 and Figure 8. The membership function of $\tilde{Q} = \tilde{3} \times \tilde{5}$ is shown in Figure 9.

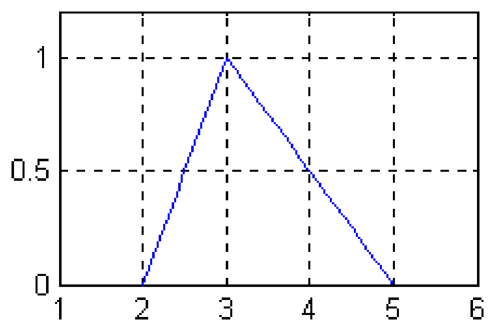


Figure 7 Membership function of $\tilde{3} = (2,3,5)$

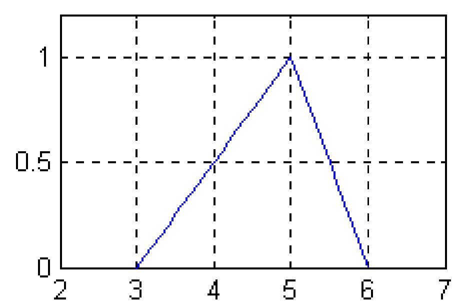


Figure 8 Membership function of $\tilde{5} = (3,5,6)$

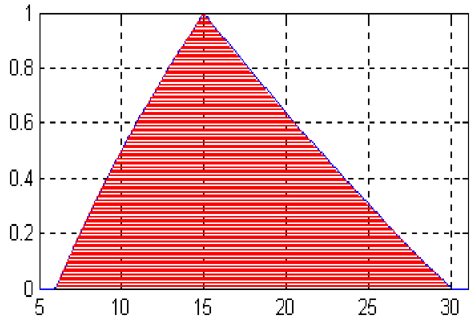


Figure 9 Membership function of $\widetilde{Q} = \widetilde{3} \times \widetilde{5}$ by analytical method

The situation is complicated when it does not satisfy $c_1 \geq b_1 \geq a_1$ and $c_2 \geq b_2 \geq a_2$ .The exact analytic equations are hard to get.

*C. Multiplication of General Fuzzy Number*

As multiplication of fuzzy number, we can also multiply scalar value to $\alpha$-cuts interval of fuzzy number. We demonstrate the method by example 2.

$\alpha$-cuts of $\widetilde{M}$ is

$$[m_\alpha^L, m_\alpha^R] = [2 - \sqrt{-\ln\alpha}, 2 + \sqrt{-\ln\alpha}]$$

$\alpha$-cuts of $\widetilde{N}$ is

$$[n_\alpha^L, n_\alpha^R] = [-2 - \sqrt{-\ln\alpha}, -2 + \sqrt{-\ln\alpha}]$$

$\alpha$-cuts of $\widetilde{Q} = \widetilde{M}(\times)\widetilde{N}$ is

$$[m_\alpha^L, m_\alpha^R] \times [n_\alpha^L, n_\alpha^R] = [q_\alpha^L, q_\alpha^R]$$

Where

$$q_\alpha^L = \min\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\} = -4 + \ln\alpha - 4\sqrt{-\ln\alpha},$$

$$q_\alpha^R = \max\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\} = \begin{cases} -\ln\alpha - 4 & 0 \leq \alpha \leq e^{-4} \\ -4 + \ln\alpha + 4\sqrt{-\ln\alpha} & e^{-4} < \alpha \leq 1 \end{cases}.$$

Substituting $q_\alpha^L = z$, we get

$$\sqrt{-\ln\alpha} = -2 + \sqrt{-z} \quad (\text{Omit } \sqrt{-\ln\alpha} < 0)$$

Therefore

$$\alpha = e^{-(2-\sqrt{-z})^2}$$

When $0 \leq \alpha \leq e^{-4}$, substituting $q_\alpha^R = z$, we get

$$\alpha = e^{-4-z}$$

When $e^{-4} < \alpha \leq 1$, substituting $q_\alpha^R = z$, we get

$$\sqrt{-\ln\alpha} = 2 - \sqrt{-z} \quad (\text{Omit } \alpha \leq e^{-4})$$

Therefore

$$\alpha = \begin{cases} e^{-(2-\sqrt{-z})^2} & z \leq 0 \\ e^{-4-z} & z > 0 \end{cases}$$

Hence membership function of $\widetilde{Q} = \widetilde{M}(\times)\widetilde{N}$ is

$$u_{\widetilde{Q}}(z) = \begin{cases} e^{-(2-\sqrt{-z})^2} & z \leq 0 \\ e^{-4-z} & z > 0 \end{cases}$$

The membership function of $\widetilde{Q} = \widetilde{3} \times \widetilde{5}$ is shown in Figure 10.

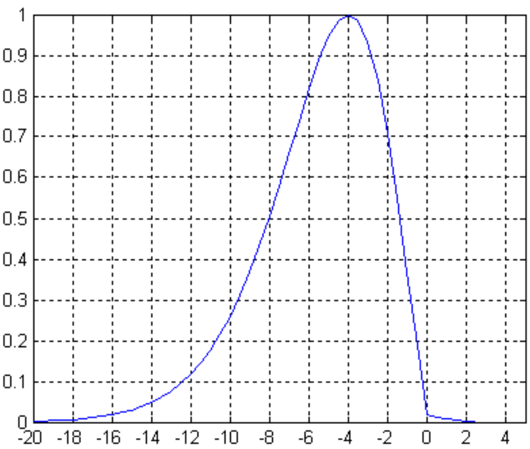


Figure 10 Membership function of $\widetilde{Q} = \widetilde{M}(\times)\widetilde{N}$ by analytical method

## V. COMPUTER DRAWING METHOD

When it does not satisfy $c_1 \geq b_1 \geq a_1$ and $c_2 \geq b_2 \geq a_2$, we can use Computer drawing method. The method based on decompose theorem is as follows:

Step 1 Set $\alpha = 0.01$;

Step 2 If $\alpha > 1$, stop. The grey sector is the membership function of $\widetilde{Q}$. Otherwise go to step 3;

Step 3 Calculate $\alpha$-cuts of $\widetilde{M}$ ($[m_\alpha^L, m_\alpha^R]$) and $\alpha$-cuts of $\widetilde{N}$ ($[n_\alpha^L, n_\alpha^R]$);

Step 4 Calculate

$$q_\alpha^L = \min\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\},$$
$$q_\alpha^R = \max\{m_\alpha^L n_\alpha^L, m_\alpha^L n_\alpha^R, m_\alpha^R n_\alpha^L, m_\alpha^R n_\alpha^R\};$$

Step 5 Set $\widetilde{Q}$ row axis and set membership function of $\widetilde{Q}$ column, then create a line from $(q_\alpha^L, \alpha)$ to $(q_\alpha^R, \alpha)$;

Step 6 $\alpha \leftarrow a + 0.01$, then go to step 2.

**Example 4** Suppose $-\widetilde{1} = (-3,-1,2)$ (in Figure 11), $\widetilde{1} = (-2,1,4)$ (in Figure 12), then the membership function of $\widetilde{Q} = -\widetilde{1} \times \widetilde{1}$ is shown in Figure 13.

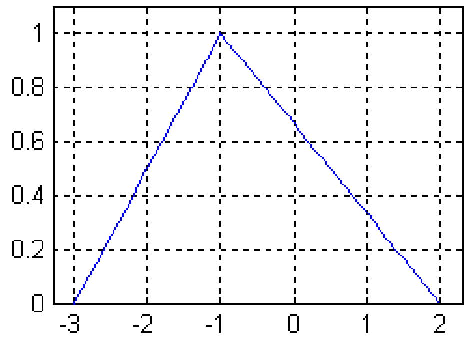


Figure11 Membership function of $-\tilde{1}=(-3,-1,2)$

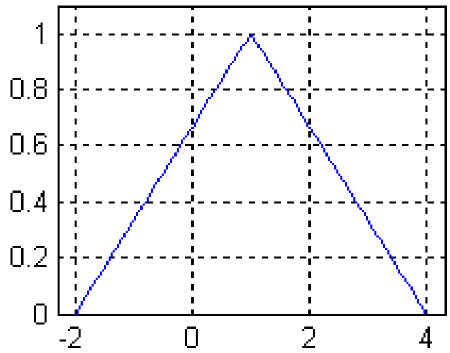


Figure 12 Membership function of $\tilde{1}=(-2,1,4)$

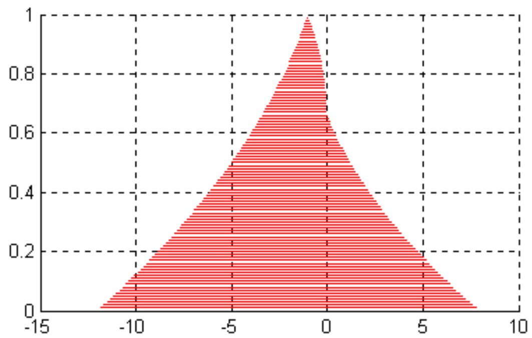


Figure 13 Membership function of $\tilde{Q}=-\tilde{1}\times\tilde{1}$ by computer drawing method

## VI. Computer Simulation Method

In computer drawing method, it need calculate $\alpha$-cuts interval. It is hard to calculate $\alpha$-cuts interval when the membership function is complicate. The computer simulation method needn't calculate $\alpha$-cuts interval. The computer simulation method is as follows:

Step 1 set $i=1$ and simulation times $N$;

Step 2 If $i>N$, stop. The grey sector is the membership function of $\tilde{Q}$. Otherwise go to step 3;

Step 3 Generate two random number $x_i$ and $y_i$ on the interval $[a_1,c_1]$ and interval $[a_2,c_2]$. Calculate $z_i=x_iy_i$, $v=\min(\mu_{\tilde{M}}(x_i),\mu_{\tilde{N}}(y_i))$;

Step 4 Create a line from $(z_i,0)$ to $(z_i,v)$, then go to step 2.

**Example 5** Suppose $\tilde{3}=(2,3,5)$ and $\tilde{5}=(3,5,6)$, then the membership function of $\tilde{Q}=\tilde{3}\times\tilde{5}$ is shown in Figure 14.

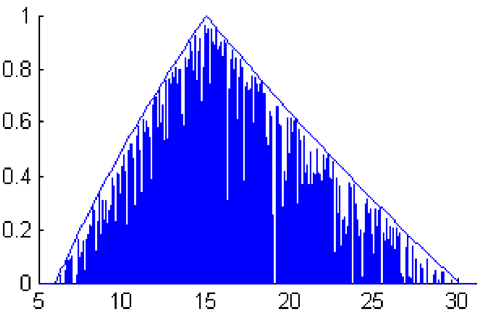


Figure 14 Membership function of $\tilde{Q}=\tilde{3}\times\tilde{5}$ by computer simulation method

## VII. Conclusions

Four methods for solving multiplication operation of two fuzzy numbers are given. Each method has advantages and disadvantages. The nonlinear programming method is a precise method also, but it only gets a membership value as given number and it is a difficult problem for solving nonlinear programming. The analytical method is most precise, but it is hard to $\alpha$-cuts interval when the membership function is complicated. The computer drawing method is simple also, but it need calculate the $\alpha$-cuts interval. The computer simulation method is the most simple, and it has wide applicability, but the membership function is rough.

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